Assignments for Math 244 for Fall 2016 (part 3)

Our next approach to solving differential equations involves the use of power series. Power series are discussed in Calculus II, but they often frighten or confuse students due to their abstract nature. They are really not that bad once you accept the fact that more thought is involved (rather than just a new skill to learn) and then become familiar with the basic idea of representing a function as an infinitely long polynomial, sort of like decimal representations for numbers. As an analogy, consider that

the fraction 10/9 **is to** the decimal $1.1111111\ldots$ **as**

the function $1/(1-x)$ **is to** the "polynomial" $1+x+x^2+x^3+x^4+x^5+x^6+\cdots$.

Writing these two expressions using series notation yields

$$\frac{10}{9}=\sum_{k=0}^{\infty}\Big(\frac{1}{10}\Big)^k \qquad \text{and} \qquad \frac{1}{1-x}=\sum_{k=0}^{\infty}x^k.$$

More generally, any real number x between 0 and 1 can be expressed as a decimal in the form $\displaystyle\sum_{k=1}^{\infty}d_k\Big(\frac{1}{10}\Big)^k$, where the digits d_k are integers between 0 and 9, inclusively. By changing the digits, we can obtain different real numbers. The important thing to notice is that all such numbers (even dramatically different ones such as $3/8$, $\sqrt{3}/2$, and $1/\pi$) have a very similar representation; all we need to know is the digits in the decimal expansion and the number is determined. Similarly, many functions (but not all) can be expressed in the form

$$\sum_{k=0}^{\infty}a_k x^k \qquad \text{or more generally} \qquad \sum_{k=0}^{\infty}a_k(x-c)^k,$$

where c and the a_k's are real numbers (so there are many more choices for the coefficients a_k besides integers 0 through 9). Thus many apparently different functions are really quite similar; we just need to change the coefficients a_k and obtain a new function. To determine a function, we need to know its corresponding sequence $\{a_k\}$ of coefficients. This property of functions can be used to turn differential equations into algebraic equations involving the coefficients a_k. We can then use algebra to find the a_k's and thus determine a function that satisfies the differential equation.

It would be in your best interest to carefully review infinite series and power series before class on Friday. We have mentioned power series once or twice and the book used them to obtain Euler's formula, but you will need to do some further work on your own. To do so, begin by reading Section 5.1 of the text and working on the problems at the end of this section. If this outline of material seems way over your head, then return to your calculus book or consult the online book Calculus Concentrate at

http://people.whitman.edu/∼gordon/math126.html

Another option is to try some Khan Academy videos on this topic if this works well with your learning style. If you decide to come to class on Friday without some working knowledge of power series, the next three weeks of differential equations will be quite a challenge for you.

for Monday, October 24

1. Read Section 5.1 of the text. If this review is not sufficient, then refer to one or more of the sources mentioned in the introduction to this set of assignments.
2. Look over problems 1–27 and do enough of them so that you begin to feel comfortable with power series. You can start with problems 2, 3, 5, 7 (find the interval of convergence on these as well), 13, 14, 16, 19, 20, 26, and 27. Can you see how problem 15 would follow almost immediately from problem 14? With a little more thought, can you see how problem 14 follows from problem 13?
3. You should know and understand the Maclaurin series for e^x, $\sin x$, and $\cos x$. This might be a good time to go back and reread page 158 to see a derivation of Euler's formula.
4. Turn in solutions for the following problems.
   a) Find the radius of convergence and the interval of convergence for $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k-1)2^k}\,(x+1)^k$.
   b) Rewrite the given sum of three power series as a single power series involving terms of the form x^k.
   $$x\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + 2\sum_{k=2}^{\infty} k a_k x^{k-1} + x\sum_{k=0}^{\infty} a_k x^k$$
   c) Find the Maclaurin series for the function f defined by $f(x) = \displaystyle\int_0^x \frac{1-e^{-t}}{t}\,dt$.
   d) Find (using familiar functions from calculus) the function represented by each power series.
   $$\sum_{k=0}^{\infty} \frac{1}{2^k k!}\,x^k, \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,x^{2k}, \qquad \sum_{k=0}^{\infty} \frac{(-2)^k}{(2k)!}\,x^{2k+1}, \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k+1)!}\,x^{2k}.$$

for Wednesday, October 26

1. Read Section 5.2 of the text.
2. Find recurrence relations for the coefficients for problems 2, 3, 5, 7, 8, and 12 in Section 5.2. You need to be patient doing the steps and pay careful attention to notation. Then find the solutions for problems 2, 5, and 7. One of the solutions to problem 7 is a simple calculus function; see if you can determine what it is. For the record, you might be able to use inspection to spot a simple solution for problem 12, then use the Wronskian method to find a second solution.
3. Turn in a solution to problem 5 from Section 5.2. Please note that I want to see all of the terms up to degree 6 for each of the two solutions; this is more detail than the answer given by the book so you need to do a bit of extra work.

for Friday, October 28

1. Continue reading Section 5.2 of the text until you fully understand the examples and techniques.
2. Do problems 5, 8, 13, and 17 in Section 5.2 USING the derivative method as discussed in class. For problem 13, one of the functions (it is the one the book calls y_1) is a simple calculus type function. It is not easy to figure this out but accept it as a challenge and see how far you can get in 15 minutes or so. Do take a look at the graphs suggested in problem 17.
3. Turn in a solution to the following problem. Solve the initial value problem

$$y'' + xy' + 2y = 0,\ y(1) = 2,\ y'(1) = -3.$$

Give the solution up to the $(x-1)^6$ term, but remember to put $\cdots$ after this term as the actual solution goes on. Then use this degree six polynomial to find an approximation for $y(1.2)$.

for Monday, October 31

1. Read Section 5.3 of the text.
2. Do problems 3, 6, 7, 10, and 15 in Section 5.3. For problem 10, consider only the case in which $\alpha = 2$ and give all of the terms of the solution that involve odd powers of x up to the 15th power, writing the coefficients as reduced fractions. It is possible to express this solution compactly using factorials; give this a try. Also, work on being able to solve problems of this type efficiently as preparation for the test on this material. Since the Chebyshev polynomials appear frequently in applied mathematics, spend a few minutes at each of the sites listed below to acquire some idea about the depth of these functions.
http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html
http://en.wikipedia.org/wiki/Chebyshev_polynomials
3. Turn in a solution to the initial value problem $y'' - e^x y' + xy = x^2 + 2x - 4, \quad y(0) = 1, \quad y'(0) = 1,$ giving the solution up through the term that involves x^6. (Do not let the fact that the differential equation is nonhomogeneous trouble you; the technique of matching up coefficients is the same. Also, be prepared for the fact that the derivatives do begin to get a bit messy as you go.) Use this solution to approximate $y(0.2)$ to the nearest thousandth. Write your work carefully so that a classmate could follow your reasoning.
4. For practice learning to "recognize" useful features of a differential equation, see if you can solve the differential equation given in problem 1 by inspection. I suggest you look at the y' and y terms as a unit and see if you can write them as a derivative.

for Wednesday, November 2

1. Read Section 5.4. Be certain to know the terminology in this section. Since we did a different derivation for solutions to Euler's equation, you can skim the first part of Section 5.4 and focus on the examples. However, note that the road to equation (9) is an application of Clairaut's Theorem from Calculus III. Spend a few minutes looking at the graphs of solutions to Euler's equation.
2. Do problems 1, 3, 4, 12, 13, 16, 18, 19, 27, 31, and 37 in Section 5.4.
3. Turn in solutions for problems 1, 13, and 18 from Section 5.4. Also, turn in a solution to the following problem. Solve the initial value problem

$$2x^2y'' + xy' - y = 0,\ y(1) = 7,\ y'(1) = 1,$$

then find the minimum value (in a simplified exact form) of the solution for $x > 0$.

for Friday, November 4

1. Read Section 5.5 of the text if necessary, primarily focusing on the technique used in the example. For practice (and good preparation for the exam), you should write the functions represented as power series in equations (19) and (21) as functions in familiar calculus terms.
2. Do problems 8, 9, and 14 in Section 5.5. You may want to do problem 9 first since the pattern for the coefficients is simpler for this differential equation and you only need to find one solution rather than two. For the record, the solution can be represented in terms of typical calculus functions; give this a try. For problem 8, just be patient, perhaps checking your work with the answers in the back of the book as you go. However, if you find yourself making too many corrections, you probably need more practice. For problem 14, I want you to just do two things. First of all, find the radius of convergence for the given power series. Secondly, show that the function represented by the power series satisfies the differential equation. This does NOT mean to solve the problem and see if you get the same answer. It DOES means that you should take the given power series, find its first two derivatives, and substitute them into the differential equation. You should get 0 of course.
3. Turn in a solution to the following problem in the spirit of Section 5.5. Consider the differential equation

$$4x^2y'' + 15x(1 + x)y' - 3y = 0.$$

Begin by finding the indicial equation and the two exponents at the singularity. Then find the solution (it should be a finite sum) for the smaller of the two exponents (it should be a negative integer). Given the form of the solution, it is easy to check to see if it works. Thus, assuming that you do not get stuck, you should know whether or not your answer is correct.

for Monday, November 7

1. We have a test covering Sections 5.1 through 5.5. You should thus review the techniques and concepts found in these sections and be certain that you can solve the problems efficiently. If you have been keeping up with the work and reviewing power series as needed, you will have little to do to prepare for the exam. More likely than not, calculators will not be allowed on the exam so plan accordingly. As I mentioned in class, I will give smaller pieces of problems in order to keep error propagation to a minimum. However, it is still important that you quickly discover the correct method for solving a problem and that you can carry out the details without a lot of trial and error. For instance, if you see a problem such as

   Find the general solution to $4x^2y'' + 23xy' - 5y = 0$

   you should recognize what to do within a few seconds. (And it better not involve series!)
2. Two exams from previous years can be found on the class website.