This is the sixth of the special assignments referred to on the syllabus. To be specific concerning what it means to work alone, you may not seek help from classmates, you may not seek help from other professors or students, you may not seek help from me (with the exception of questions concerning clarity—these should be minimal), you may not seek help from any books other than the textbook, you may not seek help from the Internet, and you may not seek help from Maple, Wolfram Alpha, or any electronic devices (except to read the textbook of course). For the record, working independently includes following the guidelines on the plagiarism form you signed when you first arrived at Whitman. Even talking briefly to someone about the problems or sharing notes is a violation of the policy. Solutions to these ten problems (worth a total of 30 points) are due by 9:00 am (Pacific time) on Monday 5/10. Write all of your solutions clearly, using complete sentences. Unless you have a valid reason for not doing so, the solutions should be formatted using LaTeX. Label your submitted file as `email_sp_06.pdf`.

Proposition: The number π is irrational.

Outline of Proof: Suppose that $\pi = p/q$, where p and q are positive integers. Let $f_0(x) = 1$ and for each positive integer n, define a function f_n by $f_n(x) = \dfrac{x^n(p-qx)^n}{n!}$. For each nonnegative integer n, let $I_n = \displaystyle\int_0^\pi f_n(x)\sin x\,dx$. Prove each of the following statements.

1. For each positive integer n, note that $f_n(x) = \dfrac{(px-qx^2)^n}{n!} = \dfrac{q^n}{n!}\,x^n(\pi-x)^n$.
2. For each positive integer n, the inequality $0 < I_n < \pi \cdot \dfrac{p^n}{n!}$ holds.
3. The sequence $\{I_n\}$ converges to 0.
4. If f is a twice differentiable function, then $\displaystyle\int \big(f(x) + f''(x)\big)\sin x\,dx = f'(x)\sin x - f(x)\cos x + C$.
5. For each integer $n \geq 2$, it follows that $I_n = -\displaystyle\int_0^\pi f_n''(x)\sin x\,dx$.
6. For each integer $n \geq 1$, we have $f_n'(x) = (p-2qx)f_{n-1}(x)$.
7. For each integer $n \geq 2$, we find that $f_n''(x) = p^2 f_{n-2}(x) - 2q(2n-1)f_{n-1}(x)$.
8. For each integer $n \geq 2$, it follows that $I_n = 2q(2n-1)I_{n-1} - p^2 I_{n-2}$.
9. The equalities $I_0 = 2$ and $I_1 = 4q$ are satisfied.
10. For each nonnegative integer n, the number I_n is a positive integer.

Since a sequence of positive integers cannot converge to 0, we have thus reached a contradiction. We conclude that π is an irrational number. ∎