**Extra Notes for Section 1.19**

Suppose that we know the following facts about positive numbers $x$ and $y$:

$$x = 10^a \quad \Leftrightarrow \quad \log_{10} x = a \qquad \text{and} \qquad y = 10^b \quad \Leftrightarrow \quad \log_{10} y = b.$$

It then follows that

$$xy = 10^a \cdot 10^b = 10^{a+b} \quad \Leftrightarrow \quad \log_{10} xy = a + b = \log_{10} x + \log_{10} y$$

and

$$\frac{x}{y} = \frac{10^a}{10^b} = 10^{a-b} \quad \Leftrightarrow \quad \log_{10} \frac{x}{y} = a - b = \log_{10} x - \log_{10} y.$$

We also see that

$$x^r = (10^a)^r = 10^{ra} \quad \Leftrightarrow \quad \log_{10} x^r = ra = r \log_{10} x$$

for any real number $r$. These specific examples using base 10 illustrate the following three important properties of logarithms ($a$, $r$, and $s$ are positive real numbers and $t$ is any real number):

$$\log_a(rs) = \log_a r + \log_a s, \qquad \log_a(r/s) = \log_a r - \log_a s, \qquad \log_a r^t = t \log_a r.$$

Since $a = a^1$ and $1 = a^0$, we know that $\log_a a = 1$ and $\log_a 1 = 0$ for any positive number $a$. We also have the two further properties:

i. $\log_a a^r = r$ for all $r \in \mathbb{R}$ (the power you must raise $a$ to in order to get $a^r$ is $r$—duh);

ii. $a^{\log_a r} = r$ for all $r > 0$ (raise $a$ to the power needed to get $r$ and you do indeed get $r$).

Really think about the somewhat flippant parenthetical comments and hopefully the equations make sense. Note that the domain is different for each equation.

We next mention two other simple, but important, properties of logarithms. In both cases, the letter $a$ represents a positive constant. For the first one, we have $e^{x \ln a} = e^{\ln a^x} = a^x$ for all real numbers $x$. For the second one, we have $\log_a x = \dfrac{\ln x}{\ln a}$ for all positive real numbers $x$. To see this, note that

$$y = \log_a x \quad \Leftrightarrow \quad a^y = x \quad \Leftrightarrow \quad \ln a^y = \ln x \quad \Leftrightarrow \quad y \ln a = \ln x \quad \Leftrightarrow \quad y = \frac{\ln x}{\ln a}.$$

Putting the two $y$ values together gives the desired result. In summary,

$$a^x = e^{x \ln a} \text{ for all } x \in \mathbb{R} \quad \text{and} \quad \log_a x = \frac{\ln x}{\ln a} \text{ for all } x > 0.$$

For a simple example of each, we have

$$3^\pi = e^{\pi \ln 3} \qquad \text{and} \qquad \log_7 100 = \frac{\ln 100}{\ln 7}.$$

**Example 1:** Suppose that $\log_a 2 = r$, $\log_a 3 = s$, and $\log_a 7 = t$, where $a$ is some positive number. Express $\log_a 84$, $\log_a 147$, $\log_a \sqrt{28}$, $\log_a(54/7)$, and $\log_a\big(2/\sqrt[3]{21}\,\big)$ in terms of $r$, $s$, and $t$.

**Solution:** Using the basic properties of logarithms, we find that

$$\begin{aligned}
\log_a 84 &= \log_a(2^2 \cdot 3 \cdot 7) = \log_a 2^2 + \log_a 3 + \log_a 7 = 2\log_a 2 + \log_a 3 + \log_a 7 = 2r + s + t; \\
\log_a 147 &= \log_a(3 \cdot 7^2) = \log_a 3 + \log_a 7^2 = \log_a 3 + 2\log_a 7 = s + 2t; \\
\log_a \sqrt{28} &= \frac{1}{2}\log_a 28 = \frac{1}{2}\log_a(2^2 \cdot 7) = \frac{1}{2}\big(\log_a 2^2 + \log_a 7\big) = \frac{1}{2}\big(2\log_a 2 + \log_a 7\big) = r + \frac{1}{2}\,t; \\
\log_a(54/7) &= \log_a(2 \cdot 3^3/7) = \log_a 2 + \log_a 3^3 - \log_a 7 = \log_a 2 + 3\log_a 3 - \log_a 7 = r + 3s - t; \\
\log_a\big(2/\sqrt[3]{21}\,\big) &= \log_a 2 - \log_a 21^{1/3} = \log_a 2 - \frac{1}{3}\big(\log_a 3 + \log_a 7\big) = r - \frac{s+t}{3}.
\end{aligned}$$

**Example 2:** Find the exact value of $x$ that satisfies $\ln(x+1) - \ln 2 = 3$.

**Solution:** Using properties of logarithms, we find that

$$\ln(x+1) - \ln 2 = 3 \quad \Leftrightarrow \quad \ln\Big(\frac{x+1}{2}\Big) = 3 \quad \Leftrightarrow \quad \frac{x+1}{2} = e^3 \quad \Leftrightarrow \quad x = 2e^3 - 1.$$

The idea is to isolate the logarithm function, then convert to exponential form.

**Example 3:** Find the exact value of $x$ that satisfies the equation $3 \cdot 2^x - 5 = 28$, then use a calculator to approximate $x$ to the nearest thousandth.

**Solution:** Performing some algebra yields

$$3 \cdot 2^x - 5 = 28 \quad \Leftrightarrow \quad 2^x = 11 \quad \Leftrightarrow \quad \ln 2^x = \ln 11 \quad \Leftrightarrow \quad x\ln 2 = \ln 11 \quad \Leftrightarrow \quad x = \frac{\ln 11}{\ln 2}.$$

The exact value of $x$ is given above; a calculator reveals that $x \approx 3.459$. For these equations, we want to isolate the exponential term and then take logarithms.

**Example 4:** Find the exact value of $x$ that satisfies the equation $\dfrac{e^x}{2+3e^x} = \dfrac{1}{4}$, then use a calculator to approximate $x$ to the nearest thousandth.

**Solution:** As with the previous example, the first step is to isolate the exponential term:

$$\frac{e^x}{2+3e^x} = \frac{1}{4} \quad \Leftrightarrow \quad 4e^x = 2 + 3e^x \quad \Leftrightarrow \quad e^x = 2 \quad \Leftrightarrow \quad x = \ln 2.$$

The exact value of $x$ that satisfies the equation is $\ln 2$, which is approximately equal to 0.693.

**Example 5:** Find the exact values of $x$ that satisfies the equation $e^{x/2} + e^{-x/2} = 20$, then use a calculator to approximate these $x$ values to the nearest thousandth.

**Solution:** Although it is in a disguised form, this is actually a quadratic equation. Performing some algebra, with the first step being multiplication by $e^{x/2}$, we find that

$$e^{x/2} + e^{-x/2} = 20 \qquad \Leftrightarrow \qquad e^{x} + 1 = 20e^{x/2} \qquad \Leftrightarrow \qquad e^{x} - 20e^{x/2} + 1 = 0.$$

Letting $y = e^{x/2}$, the last equation becomes

$$y^2 - 20y + 1 = 0 \qquad \Leftrightarrow \qquad y^2 - 20y + 100 = 99 \qquad \Leftrightarrow \qquad (y-10)^2 = 99 \qquad \Leftrightarrow \qquad y = 10 \pm \sqrt{99},$$

where we solved the quadratic equation by completing the square. It follows that

$$e^{x/2} = 10 + \sqrt{99} \qquad \Leftrightarrow \qquad x = 2\ln\bigl(10 + \sqrt{99}\bigr) \approx 5.986$$

or

$$e^{x/2} = 10 - \sqrt{99} \qquad \Leftrightarrow \qquad x = 2\ln\bigl(10 - \sqrt{99}\bigr) \approx -5.986.$$

Hence, as indicated above, there are two solutions to the equation.

**Example 6:** Suppose that a function $f$ is defined by $f(x) = Ce^{kx}$, where $C$ and $k$ are constants. Given that $f(1) = 200$ and $f(4) = 500$, find $C$ and $k$.

**Solution:** From the given information, we know that

$$f(1) = 200 \quad \Rightarrow \quad Ce^{k} = 200 \qquad \text{and} \qquad f(4) = 500 \quad \Rightarrow \quad Ce^{4k} = 500.$$

Dividing the second equation by the first equation yields

$$\frac{Ce^{4k}}{Ce^{k}} = \frac{500}{200} \quad \Leftrightarrow \quad e^{3k} = \frac{5}{2} \quad \Leftrightarrow \quad k = \frac{1}{3}\ln\left(\frac{5}{2}\right).$$

This gives us the value of $k$. Taking the cube root of each side of the middle equality in the above displayed set of equations shows that $e^{k} = \sqrt[3]{5/2}$. The equation $Ce^{k} = 200$ then tells us that

$$C = \frac{200}{e^{k}} = \frac{200}{\sqrt[3]{5/2}} = \frac{200\sqrt[3]{2}}{\sqrt[3]{5}} = \frac{200\sqrt[3]{2}}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{25}}{\sqrt[3]{25}} = 40\sqrt[3]{50}.$$

Make sure you understand all of the steps here.

## Extra Notes for Section 1.20

**Example 1:** Find and simplify the derivative of the function $f$ defined by $f(x) = e^{2x}\sin(3x)$.

**Solution:** Using the product rule, we find that

$$f'(x) = e^{2x}\big(3\cos(3x)\big) + 2e^{2x}\sin(3x) = e^{2x}\big(3\cos(3x) + 2\sin(3x)\big).$$

**Example 2:** Find and simplify the derivative of the function $g$ defined by $g(x) = \ln(x^2 + 6x + 10)$.

**Solution:** From the chain rule, it follows that

$$g'(x) = \frac{2x+6}{x^2+6x+10}.$$

**Example 3:** Find and simplify the derivative of the function $h$ defined by $h(x) = \dfrac{e^{2x}}{1+e^x}$.

**Solution:** Using the quotient rule, we find that

$$h'(x) = \frac{(1+e^x)2e^{2x} - e^{2x}(e^x)}{(1+e^x)^2} = \frac{2e^{2x} + 2e^{3x} - e^{3x}}{(1+e^x)^2} = \frac{2e^{2x} + e^{3x}}{(1+e^x)^2} = \frac{e^{2x}(2+e^x)}{(1+e^x)^2}.$$

**Example 4:** Find the exact minimum output of the function $f(x) = 12e^{-x} + e^{2x}$.

**Solution:** We first find the critical points of the function. Since

$$f'(x) = -12e^{-x} + 2e^{2x} = 2e^{-x}\big(e^{3x} - 6\big),$$

we find that $f'(x) = 0$ when $x = \frac{1}{3}\ln 6$. Noting that $f'(x) < 0$ for $x < \frac{1}{3}\ln 6$ and $f'(x) > 0$ for $x > \frac{1}{3}\ln 6$, we see that $f$ has an absolute minimum value at $x = \frac{1}{3}\ln 6$. To find this minimum value, we first note that

$$e^{-\frac{1}{3}\ln 6} = e^{\ln 6^{-1/3}} = 6^{-1/3} \qquad \text{and} \qquad e^{\frac{2}{3}\ln 6} = e^{\ln 6^{2/3}} = 6^{2/3}.$$

The absolute minimum value of the function $f$ is thus

$$f\big(\tfrac{1}{3}\ln 6\big) = 12e^{-\frac{1}{3}\ln 6} + e^{\frac{2}{3}\ln 6} = \frac{12}{6^{1/3}} + 6^{2/3} = \frac{12\cdot 6^{2/3}}{6} + 6^{2/3} = 3\sqrt[3]{36}.$$

Be certain that you follow the arithmetic in this last computation.

**Example 5:** Find the maximum and minimum outputs of $g(x) = (\ln x)/\sqrt[3]{x}$ on the interval $[1, 27]$.

**Solution:** For the record, the domain of the function $g$ is the interval $(0, \infty)$. As usual, we begin by finding the critical points of $g$, that is, the points in the domain of $g$ where $g'(x)$ does not exist or $g'(x) = 0$. By the quotient rule, we find that

$$g'(x) = \frac{x^{1/3}x^{-1} - (\ln x)\frac{1}{3}x^{-2/3}}{x^{2/3}} = \frac{x^{-2/3}\left(1 - \frac{1}{3}\ln x\right)}{x^{2/3}} = \frac{3 - \ln x}{3x^{4/3}}.$$

The only critical point occurs when $\ln x = 3$ or $x = e^3$. Since $e$ is about 2.7, we see that this value belongs to the interval $[1, 27]$. The table of values

$$g(1) = 0, \qquad g(e^3) = \frac{3}{e} \approx 1.10364, \qquad g(27) = \frac{\ln 27}{3} = \ln 3 \approx 1.09861,$$

reveals that the minimum value of $g$ is 0 and the maximum value of $g$ is $3/e$.

**Extra Notes for Section 1.21**

**Example 1:** Solve the differential equation $f'(x) = 6x^2 + 4x - 1,\ f(2) = 12$.

**Solution:** Thinking about differentiation in reverse, it is easy to see that

$$f(x) = 2x^3 + 2x^2 - x + C,$$

where $C$ is some constant. Since $f(2) = 12$, we find that

$$12 = f(2) = 16 + 8 - 2 + C = 22 + C \quad \Rightarrow \quad C = -10.$$

The function $f(x) = 2x^3 + 2x^2 - x - 10$ satisfies the differential equation.

**Example 2:** Find a function $g$ that satisfies the conditions $g'(x) = 2x - \dfrac{1}{x},\ g(1) = 4$.

**Solution:** Mentally searching our knowledge of derivatives, we find that

$$g(x) = x^2 - \ln x + C,$$

where $C$ is some constant. (Remember that you can always, as in ALWAYS, check your answers by taking a derivative.) Since $g(1) = 4$, it follows that

$$4 = g(1) = 1 - 0 + C = 1 + C \quad \Rightarrow \quad C = 3.$$

The function $g(x) = x^2 - \ln x + 3$ satisfies the given conditions.

**Example 3:** Find a function $h$ that satisfies the conditions $h'(t) = \sin t + 4\cos(2t),\ h(\pi) = -1$.

**Solution:** Given our knowledge of the derivatives of the trigonometric functions, we see that

$$h(t) = -\cos t + 2\sin(2t) + C,$$

where $C$ is some constant. Note how the coefficient of the second term must change to compensate for the chain rule. The fact that $h(\pi) = -1$ yields

$$-1 = h(\pi) = 1 + 0 + C \quad \Rightarrow \quad C = -2.$$

The function $h(t) = -\cos t + 2\sin(2t) - 2$ satisfies the given conditions.

**Example 4:** Solve the differential equations $f'(x) = -5f(x),\ f(0) = 40$ and $g'(x) = \frac{1}{5}g(x),\ g(0) = 10$.

**Solution:** Referring to Theorem 1.20, we find that $f(x) = 40e^{-5x}$ and $g(x) = 10e^{x/5}$. To check the first one, note that

$$f'(x) = 40e^{-5x}(-5) = -5f(x) \quad \text{and} \quad f(0) = 40e^0 = 40.$$

**Example 5:** Find a function $F$ that satisfies the conditions $F'(x) = x\sqrt{x^2+4},\ F(0) = 2$.

**Solution:** This differential equation is a little trickier than the others. Noting that differentiation reduces the exponent by one, we might try guessing that $F(x) = (x^2+4)^{3/2}$. We then have

$$F'(x) = \frac{3}{2}(x^2+4)^{1/2}(2x) = 3x\sqrt{x^2+4},$$

which is close but not quite correct as we are off by a factor of 3. Adjusting for this factor, it follows that $F(x) = \frac{1}{3}(x^2+4)^{3/2} + C$ is the correct function. Since $F(0) = 2$, we find that

$$2 = F(0) = \frac{1}{3} \cdot 4^{3/2} + C = \frac{8}{3} + C \quad \Rightarrow \quad C = -\frac{2}{3}.$$

Therefore, the function $F(x) = \frac{1}{3}(x^2+4)^{3/2} - \frac{2}{3}$ satisfies the given conditions.

**Example 6:** Solve the differential equation $A'(t) = 40 + 4A(t),\ A(0) = 20$.

**Solution:** Performing some algebra, we can write the differential equation as

$$A'(t) = 40 + 4A(t) = 4(A(t)+10) \quad \text{or} \quad (A(t)+10)' = 4(A(t)+10).$$

We can now use Theorem 1.20. You should be able to do this directly, but we will add another step for this solution. Let $B(t) = A(t) + 10$. Referring to the information about the function $A(t)$, we find that $B'(t) = 4B(t),\ B(0) = A(0) + 10 = 30$. By Theorem 1.20, it follows that $B(t) = 30e^{4t}$. Using the definition of $B(t)$, we find that the function $A(t) = 30e^{4t} - 10$ satisfies the differential equation.

**Example 7:** Solve the differential equation $S'(t) = 100 - 0.4S(t),\ S(0) = 200$.

**Solution:** We use similar steps to those in the previous problem. The differential equation is equivalent to

$$S'(t) = -0.4(S(t)-250) \quad \text{or} \quad (S(t)-250)' = -0.4(S(t)-250).$$

Since $S(0) - 250 = 200 - 250 = -50$, Theorem 1.20 tells us that $S(t) - 250 = -50e^{-0.4t}$. Hence, the function $S(t) = 250 - 50e^{-0.4t}$ satisfies the differential equation.

**Extra Notes for Section 1.22**

**Example 1:** A certain radioactive element decays at a rate proportional to its current mass. Initially there were 500 grams of the element and two hours later there were 480 grams. When will only 100 grams of the element exist?

**Solution:** Let $A(t)$ be the number of grams of the radioactive element after $t$ hours. We are given that $A(0) = 500$ and $A(2) = 480$, and we want to find a value of $t$ when $A(t) = 100$. Since the rate of change of $A(t)$ is proportional to $A(t)$, there is a positive constant $k$ such that $A'(t) = -kA(t)$. (Typically, parameters such as decay constants are assumed to be positive. We need the minus sign since the function $A(t)$ is decreasing and thus has a negative derivative.) Referring to Theorem 1.20, we find that $A(t) = 500e^{-kt}$. To find the value of $k$, we use the fact that $A(2) = 480$:

$$A(2) = 480 \quad \Rightarrow \quad 500e^{-2k} = 480 \quad \Rightarrow \quad e^{2k} = \frac{25}{24} \quad \Rightarrow \quad k = \frac{1}{2}\ln\Big(\frac{25}{24}\Big).$$

Rather than find a decimal approximation for $k$ at this stage (which could lead to round-off errors), we just continue to use the letter $k$ for the constant but the above formula can be used at any time to find $k$. To answer the question, we need to solve the equation $A(t) = 100$ for $t$:

$$A(t) = 100 \quad \Rightarrow \quad 500e^{-kt} = 100 \quad \Rightarrow \quad e^{kt} = 5 \quad \Rightarrow \quad t = \frac{\ln 5}{k} = \frac{2\ln 5}{\ln(25/24)} \approx 78.8515.$$

Hence, there will be 100 grams of the element left about 78.85 hours after there were 500 grams.

**Example 2:** The velocity of a falling object satisfies the differential equation $v'(t) = 32 - 2v(t)$, where $v$ is measured in feet per second (with the downward direction being positive) and $t$ is measured in seconds. Suppose that the initial velocity of the object is 100 feet per second in the downward direction. Determine the time when the object is falling at a rate of 18 feet per second.

**Solution:** From the given information about the falling object, we need to solve the differential equation $v'(t) = 32 - 2v(t),\ v(0) = 100$. We can solve this differential equation using the techniques we learned in the last section. Noting that

$$(v(t) - 16)' = -2(v(t) - 16) \quad \text{and} \quad v(0) - 16 = 84,$$

we find that $v(t) - 16 = 84e^{-2t}$ and thus $v(t) = 16 + 84e^{-2t}$. To answer the question posed in the problem, we must solve the equation $v(t) = 18$ for $t$. In this case, we have (fill in the missing steps)

$$16 + 84e^{-2t} = 18 \quad \Rightarrow \quad e^{2t} = 42 \quad \Rightarrow \quad t = \frac{1}{2}\ln 42 \approx 1.8688.$$

Therefore, the object is falling at a speed of 18 feet per second after about 1.87 seconds.

## Extra Notes for Section 1.24

**Example 1:** Evaluate $\lim_{x\to\infty} \dfrac{2x^4 - 4x + 10}{3x^4 + 5x^3}$.

**Solution:** Since both the numerator and the denominator of the function grow indefinitely large as $x \to \infty$, the limit has the indeterminate form $\infty/\infty$. We identify the highest power of $x$ in the denominator and then use this value to multiply the fraction by the number 1 in a useful form:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^4 - 4x + 10}{3x^4 + 5x^3} &= \lim_{x\to\infty} \left( \frac{2x^4 - 4x + 10}{3x^4 + 5x^3} \cdot \frac{1/x^4}{1/x^4} \right) && \text{(multiply by 1)} \\
&= \lim_{x\to\infty} \frac{2 - \dfrac{4}{x^3} + \dfrac{10}{x^4}}{3 + \dfrac{5}{x}} && \text{(perform some algebra)} \\
&= \frac{2 - 0 + 0}{3 + 0} = \frac{2}{3}. && \text{(recognize the simple limits)}
\end{aligned}$$

**Example 2:** Evaluate $\lim_{x\to\infty} \dfrac{4x - 7}{\sqrt{6x^2 - 2x + 3}}$.

**Solution:** This limit also has the indeterminate form $\infty/\infty$. Once again, we identify the highest power of $x$ in the denominator. Since the term $x^2$ appears in a square root, the highest power of $x$ in the denominator is $x$. Hence,

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x - 7}{\sqrt{6x^2 - 2x + 3}} &= \lim_{x\to\infty} \left( \frac{4x - 7}{\sqrt{6x^2 - 2x + 3}} \cdot \frac{1/x}{1/x} \right) && \text{(multiply by 1)} \\
&= \lim_{x\to\infty} \frac{4 - \dfrac{7}{x}}{\sqrt{6 - \dfrac{2}{x} + \dfrac{3}{x^2}}} && \text{(perform some algebra)} \\
&= \frac{4 - 0}{\sqrt{6 - 0 + 0}} = \frac{4}{\sqrt{6}}. && \text{(recognize the simple limits)}
\end{aligned}$$

Note that $1/x$ enters the square root as $1/x^2$.

**Example 3:** Evaluate $\lim_{x\to 5^+} \dfrac{2x - 9}{x^2 - 11x + 30}$.

**Solution:** The function that appears in this limit has the form $1/0$ as $x$ approaches 5. We thus know that an infinite limit is involved and all we need to do is determine if the function is going to $\infty$ or $-\infty$. Factoring the denominator yields

$$\frac{2x - 9}{x^2 - 11x + 30} = \frac{2x - 9}{(x - 5)(x - 6)}.$$

For $x$ near 5 but a little bigger than 5 (recall what $x \to 5^+$ means), we see that $x - 5$ is a small positive number and $x - 6$ is a negative number (very close to $-1$). It follows that

$$\lim_{x\to 5^+} \frac{2x - 9}{x^2 - 11x + 30} = -\infty.$$

**Example 4:** Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+6x+90}-x\right)$.

**Solution:** This limit has the form $\infty-\infty$, which is indeterminate. The first step is to multiply by the conjugate and then "proceed as usual."

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+6x+90}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+6x+90}-x\right)\cdot\frac{\sqrt{x^2+6x+90}+x}{\sqrt{x^2+6x+90}+x} && \text{(multiply by 1)}\\
&= \lim_{x\to\infty}\frac{(x^2+6x+90)-x^2}{\sqrt{x^2+6x+90}+x} && \text{(perform some algebra)}\\
&= \lim_{x\to\infty}\frac{6x+90}{\sqrt{x^2+6x+90}+x} && \text{(simplify)}\\
&= \lim_{x\to\infty}\frac{6x+90}{\sqrt{x^2+6x+90}+x}\cdot\frac{1/x}{1/x} && \text{(multiply by 1)}\\
&= \lim_{x\to\infty}\frac{6+\frac{90}{x}}{\sqrt{1+\frac{6}{x}+\frac{90}{x^2}}+1} && \text{(perform some algebra)}\\
&= \frac{6+0}{\sqrt{1}+1}=3. && \text{(recognize the simple limits)}
\end{aligned}
$$

**Example 5:** Determine all of the asymptotes, both vertical and horizontal, for $f(x)=\dfrac{3(x^2+7x+6)}{x^2-36}$.

**Solution:** Using the ideas presented in the previous examples, we easily see that

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{3(x^2+7x+6)}{x^2-36}=3 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x)=3.$$

This shows that $y=3$ is a horizontal asymptote for the function $f$. To determine vertical asymptotes, we factor the numerator and denominator of $f$ to obtain

$$f(x)=\frac{3(x+6)(x+1)}{(x-6)(x+6)}=\frac{3(x+1)}{x-6},$$

where the last step is valid for all $x\neq-6$. It follows that $x=6$ is a vertical asymptote of $f$. The function $f$ is not defined at $-6$ but it does have a limit at $-6$:

$$\lim_{x\to-6} f(x)=\lim_{x\to-6}\frac{3(x+1)}{x-6}=\frac{3(-5)}{-12}=\frac{5}{4}.$$

As a result, there is no vertical asymptote at this point. The function $f$ thus has two asymptotes, namely, the lines $y=3$ and $x=6$.

**Extra Notes for Section 1.25**

**Example 1:** Evaluate $\lim_{x\to 0} \dfrac{\sin x + \sin(2x)}{4x + 1 - \cos(3x)}$.

**Solution:** Since the limit has the form 0/0, we can apply L'Hôpital's Rule:

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin x + \sin(2x)}{4x + 1 - \cos(3x)} && \text{(0/0 form)} \\
&= \lim_{x\to 0} \frac{\cos x + 2\cos(2x)}{4 + 3\sin(3x)} && \text{(apply L'Hôpital's Rule)} \\
&= \frac{1+2}{4+0} = \frac{3}{4}. && \text{(evaluate the limit)}
\end{aligned}$$

**Example 2:** Evaluate $\lim_{x\to 0} (1 + \tan(3x))^{1/x}$.

**Solution:** This limit has the indeterminate form $1^\infty$. Since an exponential indeterminate form is involved, we must first solve a different problem. Consider the limit $\lim_{x\to 0} \ln\big(1 + \tan(3x)\big)^{1/x}$. Using the properties of logarithms, we find that

$$\begin{aligned}
\lim_{x\to 0} \ln\big(1 + \tan(3x)\big)^{1/x} &= \lim_{x\to 0} \frac{1}{x} \cdot \ln\big(1 + \tan(3x)\big) && \text{(property of ln)} \\
&= \lim_{x\to 0} \frac{\ln\big(1 + \tan(3x)\big)}{x} && \text{(0/0 form)} \\
&= \lim_{x\to 0} \frac{3\sec^2(3x)}{1 + \tan(3x)} && \text{(apply L'Hôpital's Rule)} \\
&= \frac{3}{1+0} = 3. && \text{(evaluate the limit)}
\end{aligned}$$

We can now find the original limit—call it $L$—by noting that we have just shown that $\ln L = 3$. This means that $L = e^3$. Thus $\lim_{x\to 0} (1 + \tan(3x))^{1/x} = e^3$.

**Example 3:** Evaluate $\lim_{x\to\infty} \big(xe^{-7/x} - x\big)$.

**Solution:** This limit has the indeterminate form $\infty - \infty$ so we must do some algebra before applying L'Hôpital's Rule: (multiplying by $x$ is the same as dividing by $1/x$)

$$\begin{aligned}
\lim_{x\to\infty} \big(xe^{-7/x} - x\big) &= \lim_{x\to\infty} \frac{e^{-7/x} - 1}{1/x} && \text{(perform some algebra)} \\
&= \lim_{x\to\infty} \frac{e^{-7/x} \cdot \frac{7}{x^2}}{-\frac{1}{x^2}} && \text{(apply L'Hôpital's Rule)} \\
&= \lim_{x\to\infty} -7e^{-7/x} && \text{(simplify)} \\
&= -7e^0 = -7. && \text{(evaluate the limit)}
\end{aligned}$$

**Extra Notes for Section 1.26**

**Example 1:** Find the point $c$ guaranteed by the Mean Value Theorem for the function $f(x) = x^3 - x^2 + 3x$ on the interval $[1, 3]$.

**Solution:** It is easy to see that the hypotheses of the MVT are satisfied by the function $f$ on the given interval. For this particular function and interval, the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{becomes} \qquad 3c^2 - 2c + 3 = \frac{(27 - 9 + 9) - (1 - 1 + 3)}{3 - 1} = \frac{27 - 3}{2} = 12.$$

Solving this equation for $c$, we find that (using the quadratic formula)

$$3c^2 - 2c - 9 = 0 \quad \Rightarrow \quad c = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-9)}}{6} = \frac{2 \pm \sqrt{112}}{6} = \frac{2 \pm 4\sqrt{7}}{6} = \frac{1 \pm 2\sqrt{7}}{3}.$$

Since the point $c$ lies in the interval $(1, 3)$, we must choose the plus sign. Hence, the point $c$ guaranteed by the Mean Value Theorem is $\dfrac{1 + 2\sqrt{7}}{3}$.

**Example 2:** Prove that the function $f(x) = 11x - 3\cos(\pi x)$ is increasing on $\mathbb{R}$.

**Solution:** Recall that $-1 \le \sin x \le 1$ for all $x$. Since

$$f'(x) = 11 + 3\pi \sin(\pi x) \ge 11 - 3\pi > 0$$

for all real numbers, it follows that $f$ is increasing on $\mathbb{R}$.

**Example 3:** Prove that the function $g(x) = x^5 - 20x^3 + 200x$ is increasing on $\mathbb{R}$.

**Solution:** Completing the square to simplify the derivative, we find that

$$g'(x) = 5x^4 - 60x^2 + 200 = 5(x^4 - 12x^2 + 40) = 5\big((x^2 - 6)^2 + 4\big) > 0$$

for all real numbers. Hence, the function $g$ is increasing on $\mathbb{R}$.

**Example 4:** Suppose that $f$ is a differentiable function and that the equation $f(x) = 0$ has four distinct solutions. Prove that the equation $f'(x) = 0$ has at least three distinct solutions.

**Solution:** Suppose that $a$, $b$, $c$, and $d$ are the four solutions to $f(x) = 0$ with $a < b < c < d$. By the MVT, there exist points $s \in (a, b)$, $t \in (b, c)$, and $u \in (c, d)$ such that

$$f'(s) = \frac{f(b) - f(a)}{b - a} = 0, \qquad f'(t) = \frac{f(c) - f(b)}{c - b} = 0, \qquad \text{and} \qquad f'(u) = \frac{f(d) - f(c)}{d - c} = 0.$$

The points $s$, $t$, and $u$ are thus three distinct solutions to the equation $f'(x) = 0$.

## Extra Notes for Section 1.30

**Example 1:** Use the linear, quadratic, cubic, and quartic polynomial approximations to $\sqrt[3]{x}$ centered at $c = 1$ to estimate the values of $\sqrt[3]{1.04}$ and $\sqrt[3]{1.2}$.

**Solution:** Let $f(x) = \sqrt[3]{x}$. Referring to the formulas given in the section, we begin by finding the first four derivatives of $f$ and evaluating them at the center $c = 1$. After doing so, we divide each of these numbers by the corresponding factorial to determine the appropriate coefficients.

$$
\begin{aligned}
f(x) &= x^{1/3}; & f(1) &= 1; & a_0 &= f(1) = 1;\\
f'(x) &= \frac{1}{3}\,x^{-2/3}; & f'(1) &= \frac{1}{3}; & a_1 &= \frac{f'(1)}{1!} = \frac{1}{3};\\
f''(x) &= -\frac{2}{9}\,x^{-5/3}; & f''(1) &= -\frac{2}{9}; & a_2 &= \frac{f''(1)}{2!} = -\frac{1}{9};\\
f'''(x) &= \frac{10}{27}\,x^{-8/3}; & f'''(1) &= \frac{10}{27}; & a_3 &= \frac{f'''(1)}{3!} = \frac{5}{81};\\
f^{(4)}(x) &= -\frac{80}{81}\,x^{-11/3}; & f^{(4)}(1) &= -\frac{80}{81}; & a_4 &= \frac{f^{(4)}(1)}{4!} = -\frac{10}{243}.
\end{aligned}
$$

It follows that the linear, quadratic, cubic, and quartic polynomials that best approximate $\sqrt[3]{x}$ near the point 1 are given by

$$
\begin{aligned}
\ell(x) &= 1 + \frac{1}{3}(x-1);\\
q(x) &= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2;\\
c(x) &= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3;\\
Q(x) &= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3 - \frac{10}{243}(x-1)^4;
\end{aligned}
$$

respectively. As a reminder, the function $c(x)$ (to take one example) is a cubic polynomial that has the same derivatives as $f(x)$ at $c = 1$ up to the third derivative, that is

$$f(1) = c(1), \quad f'(1) = c'(1), \quad f''(1) = c''(1), \quad f'''(1) = c'''(1).$$

You should check that this is indeed the case. We can then use these polynomials to approximate the values of $f(x)$ for inputs $x$ near 1. The key idea is that the polynomial functions only require addition and multiplication to evaluate and this is all a computer can be programmed to do. When you ask a calculator for the value of $\sqrt[3]{1.04}$, it uses some polynomial to return a value. Referring to the above functions, we find that

$$
\begin{aligned}
\ell(1.04) &\approx 1.01333333; & \ell(1.2) &\approx 1.06666667;\\
q(1.04) &\approx 1.01315556; & q(1.2) &\approx 1.06222222;\\
c(1.04) &\approx 1.01315951; & c(1.2) &\approx 1.06271649;\\
Q(1.04) &\approx 1.01315940; & Q(1.2) &\approx 1.06265021;\\
\sqrt[3]{1.04} &\approx 1.01315940; & \sqrt[3]{1.2} &\approx 1.06265857;
\end{aligned}
$$

Note that the approximations improve as the degree of the polynomial increases and that the approximations are less accurate as you move away from the center.

**Extra Notes for Section 1.32**

**Example 1:** Find $dy/dx$ both explicitly and implicitly for the function $y$ that satisfies $y^2-4xy+x^2-x=0$.

**Solution:** We first find the functions that satisfy the given equation. Completing the square, we find that

$$y^2-4xy+x^2-x=0 \quad\Leftrightarrow\quad y^2-4xy+4x^2=-x^2+x+4x^2 \quad\Leftrightarrow\quad (y-2x)^2=3x^2+x.$$

We thus obtain the two functions (followed by their derivatives)

$$y=2x+\sqrt{3x^2+x} \qquad\Rightarrow\qquad \frac{dy}{dx}=2+\frac{6x+1}{2\sqrt{3x^2+x}}=2+\frac{6x+1}{2(y-2x)}=\frac{4y-2x+1}{2y-4x}$$

and

$$y=2x-\sqrt{3x^2+x} \qquad\Rightarrow\qquad \frac{dy}{dx}=2-\frac{6x+1}{2\sqrt{3x^2+x}}=2+\frac{6x+1}{2(y-2x)}=\frac{4y-2x+1}{2y-4x}.$$

To get the second form for each derivative listed above, we used the fact that $y-2x=\sqrt{3x^2+x}$ for the first function and $y-2x=-\sqrt{3x^2+x}$ for the second function. Hence, we have found the derivatives of the two functions explicitly. For implicit differentiation, we simply assume that $y$ is a function of $x$ without solving for $y$ and differentiate accordingly. This gives

$$\frac{d}{dx}\big(y^2-4xy+x^2-x\big)=\frac{d}{dx}(0) \quad\Rightarrow\quad 2y\cdot\frac{dy}{dx}-\left(4x\cdot\frac{dy}{dx}+4y\right)+2x-1=0 \quad\Rightarrow\quad \left(2y-4x\right)\frac{dy}{dx}=-2x+1.$$

It follows that

$$\frac{dy}{dx}=\frac{4y-2x+1}{2y-4x},$$

the same value we obtained with the explicit method after substituting for the square root term.

**Example 2:** Find an equation for the line tangent to the curve defined by $x^4+y^3=4xy^2+64$ at $(-1,3)$.

**Solution:** We first check that the given point is on the curve:

$$(-1)^4+3^3=28 \qquad\text{and}\qquad 4(-1)3^2+64=28.$$

Since the coordinates of the point satisfy the equation, we know it is on the curve. Next, we use implicit differentiation to find $dy/dx$.

$$\frac{d}{dx}\left(x^4+y^3\right)=\frac{d}{dx}\left(4xy^2+64\right) \quad\Rightarrow\quad 4x^3+3y^2\cdot\frac{dy}{dx}=8xy\cdot\frac{dy}{dx}+4y^2 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{4y^2-4x^3}{3y^2-8xy}.$$

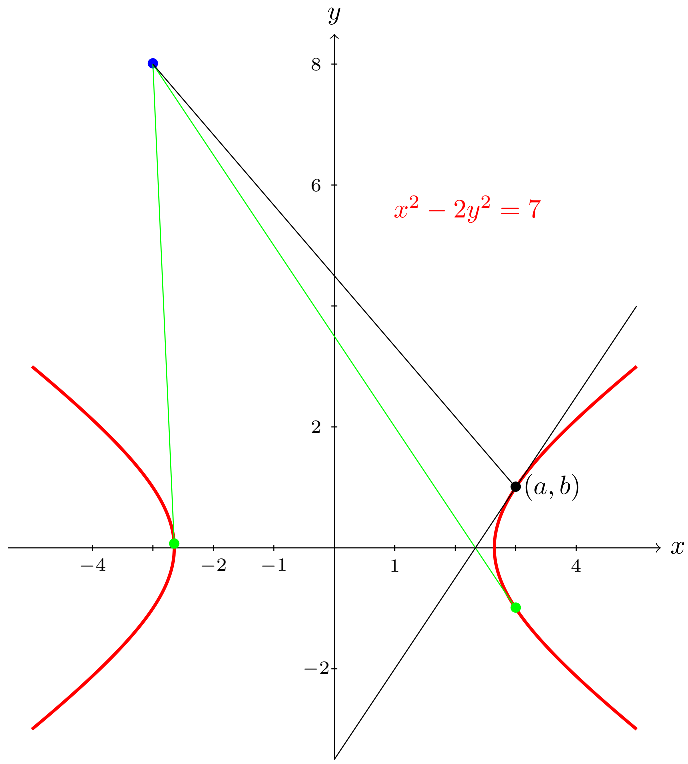
(Be certain you understand the algebra used to solve for $dy/dx$.) We then have

$$\left.\frac{dy}{dx}\right|_{(-1,3)}=\frac{4(3)^2-4(-1)^3}{3(3)^2-8(-1)3}=\frac{36+4}{27+24}=\frac{40}{51}.$$

It follows that an equation for the tangent line to the curve at the point $(-1,3)$ is $y-3=\frac{40}{51}(x+1)$.

**Example 3:** Find a point on the curve $x^2 - 2y^2 = 7$ for which the tangent line to the curve passes through the point$(-3, 8)$.

**Solution:** The graph of this function is a hyperbola; it is shown in red below with the point $(-3, 8)$ denoted by a blue dot.




Consider a typical point $(a, b)$ on the graph of the hyperbola and note that $a^2 - 2b^2 = 7$ since it is on the curve. We can sketch two lines at each such point: the tangent line to the curve and the line from the point to $(-3, 8)$. These are shown in black. We are seeking the points on the curve where the two lines are the same; the green dots illustrate these points. Using implicit differentiation, we find that

$$2x - 4y \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{2y}.$$

In order for the tangent line at $(a, b)$ to go through the point $(-3, 8)$, the slope of the tangent line must equal the slope of the line that goes to the point $(-3, 8)$. We thus have (recall that $a^2 - 2b^2 = 7$)

$$\frac{a}{2b} = \frac{b-8}{a+3} \quad \Rightarrow \quad a^2 + 3a = 2b^2 - 16b \quad \Rightarrow \quad 7 = a^2 - 2b^2 = -3a - 16b \quad \Rightarrow \quad a = \frac{7+16b}{-3}.$$

Substituting this value for $a$ into the equation $a^2 - 2b^2 = 7$ yields

$$\frac{(7+16b)^2}{9} - 2b^2 = 7 \quad \Rightarrow \quad 49 + 14 \cdot 16b + 256b^2 - 18b^2 = 63 \quad \Rightarrow \quad 238b^2 + 14 \cdot 16b - 14 = 0$$

$$\Rightarrow \quad 17b^2 + 16b - 1 = 0 \quad \Rightarrow \quad (b+1)(17b-1) = 0.$$

Hence, the values for $b$ are

$$b = -1 \text{ with } a = \frac{7+16(-1)}{-3} = 3 \quad \text{and} \quad b = \frac{1}{17} \text{ with } a = \frac{7+16(1/17)}{-3} = \frac{135/17}{-3} = -\frac{45}{17}.$$

The points with the requested properties are $(3, -1)$ and $\left(-\frac{45}{17}, \frac{1}{17}\right)$, which are reasonable answers given the graph of the curve shown above.