Extra Notes for Section 1.1

Example 1: Find an equation for the line that goes through the point $(1, 2)$ and is perpendicular to the line through the points $(-1, 4)$ and $(6, 19)$.

Solution: The slope of the line through the points $(-1, 4)$ and $(6, 19)$ is

$$\frac{19-4}{6-(-1)} = \frac{15}{7}.$$

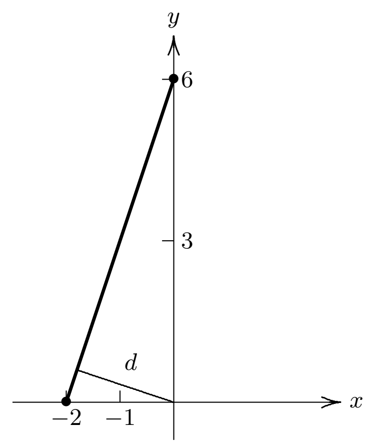
Any line perpendicular to this line must have slope $-7/15$. An equation for the perpendicular line that goes through the point $(1, 2)$ is thus

$$y - 2 = -\frac{7}{15}(x-1) \qquad \text{or} \qquad 7x + 15y = 37 \qquad \text{or} \qquad y = -\frac{7}{15}x + \frac{37}{15}.$$

Any of the forms for the equation of the line is acceptable. ■

Example 2: Find the distance from the origin to the line through the points $(-2, 0)$ and $(0, 6)$.

Solution: The distance from a point to a line refers to the perpendicular distance from the point to the line. The problem is illustrated in the following figure.



We need to find the distance of the line segment labeled d in the figure. One option is to find the slope of the line ℓ through the points $(-2, 0)$ and $(0, 6)$ (the slope is 3), obtain an equation for the line through the origin that is perpendicular to ℓ (the equation is $y = -x/3$), and find the point of intersection of this line with the line ℓ (the point is $(-1.8, 0.6)$). The distance from the line ℓ to the origin is then the distance from this point to the origin. The requested distance is thus $\frac{3}{5}\sqrt{10}$.

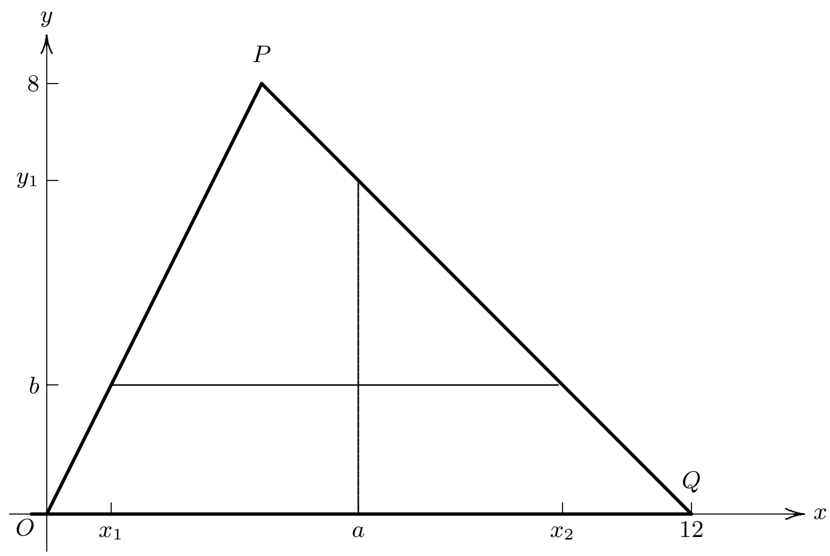
However, there is an easier way to proceed. Consider the triangle T with vertices $(0, 0)$, $(-2, 0)$, and $(0, 6)$. We can compute the area of this right triangle in two different ways. Using the 'easy' way, we find that the area is $\frac{1}{2} \cdot 2 \cdot 6$. With the hypotenuse as the base, the area is given by $\frac{1}{2}\sqrt{40} \cdot d$. Since the area of the triangle is the same no matter how it is computed, it follows that

$$2 \cdot 6 = \sqrt{40} \cdot d \qquad \text{or} \qquad d = \frac{2 \cdot 6}{2 \cdot \sqrt{10}} = \frac{3}{5}\sqrt{10}.$$

Hence, the distance from the origin to the line through the points $(-2, 0)$ and $(0, 6)$ is $\frac{3}{5}\sqrt{10}$. ■

Example 3: Let T be the triangle with vertices $(0,0)$, $(4,8)$, and $(12,0)$. Find both a horizontal line and a vertical line that divides T into two parts of equal area.

Solution: We first sketch a graph in order to visualize what we are trying to accomplish:



We want to find a horizontal line $y = b$ and a vertical line $x = a$ that divide $\triangle OPQ$ into two parts of equal area. Before proceeding, we make note of the following facts:

a. The area of triangle T is 48.

b. The equation of the line through points O and P is $y = 2x$.

c. The equation of the line through points P and Q is $x + y = 12$.

It then follows that $x_1 = b/2$, $x_2 = 12 - b$, and $y_1 = 12 - a$, where the positions x_1, x_2, and y_1 are indicated in the figure.

We first find the vertical line $x = a$ that splits T into two parts of equal area. The area of the triangle to the right of the green line is $\frac{1}{2}(12-a)y_1$. Since $y_1 = 12 - a$, we find that

$$\frac{1}{2}(12-a)(12-a) = 24 \qquad \text{and thus} \qquad a = 12 - \sqrt{48} = 12 - 4\sqrt{3}.$$

The area of the triangle above the blue line is $\frac{1}{2}(x_2 - x_1)(8-b)$. In order for this area to be half of the area of T, we must have

$$\frac{1}{2}\left(12 - b - \frac{b}{2}\right)(8-b) = 24 \qquad \Leftrightarrow \qquad \frac{1}{2}\left(\frac{24-3b}{2}\right)(8-b) = 24 \qquad \Leftrightarrow \qquad (8-b)^2 = 32.$$

It then follows that $b = 8 - 4\sqrt{2}$. (If you are familiar with similar triangles, it is possible to obtain this value in another way.) Hence, the horizontal line $y = 8 - 4\sqrt{2}$ divides T into two parts of equal area, and the vertical line $x = 12 - 4\sqrt{3}$ divides T into two parts of equal area. ■

Extra Notes for Section 1.2

Example 1: Find the rational number represented by the repeating decimal $0.247474747\ldots$.

Solution: Let $x = 0.247474747\ldots$. We then have

$$1000x - 10x = 247.47474747\ldots - 2.47474747\ldots = 245.$$

It follows that $x = 245/990 = 49/198$. Note that the multiples of x are chosen so that the repeating part of the decimal disappears upon subtraction. ∎

Example 2: Find the domain of the function f defined by $f(x) = \dfrac{x+2}{\sqrt{20 - x - x^2}}$.

Solution: We need to avoid taking the square root of a negative number and we need to avoid dividing by 0. Factoring the polynomial in the square root yields

$$20 - x - x^2 = (5 + x)(4 - x).$$

This quantity is positive as long as $x > -5$ and $x < 4$. The domain of the function f is the interval $(-5, 4)$. For the record, the domain of the function g defined by $g(x) = \sqrt{20 - x - x^2}$ is the closed interval $[-5, 4]$. In this case, the 0 values in the square root are not a problem. ∎

Example 3: For the functions f and g defined by $f(x) = x^2 + x$ and $g(x) = x - 3$, respectively, find formulas for the functions $f \circ g$, $g \circ f$, and $f \circ f$.

Solution: Using the definition of composition of functions, we find that

$$(f \circ g)(x) = f(g(x)) = (x-3)^2 + (x-3) = (x^2 - 6x + 9) + (x - 3) = x^2 - 5x + 6;$$
$$(g \circ f)(x) = g(f(x)) = (x^2 + x) - 3 = x^2 + x - 3;$$
$$(f \circ f)(x) = f(f(x)) = (x^2 + x)^2 + (x^2 + x) = (x^4 + 2x^3 + x^2) + (x^2 + x) = x^4 + 2x^3 + 2x^2 + x.$$

Note that $f \circ g$ does not equal $g \circ f$, that is, the operation of composition is not commutative. ∎

Example 4: Express the set $\{x \in \mathbb{R} : 4 + |3x - 1| < 11\}$ as an interval.

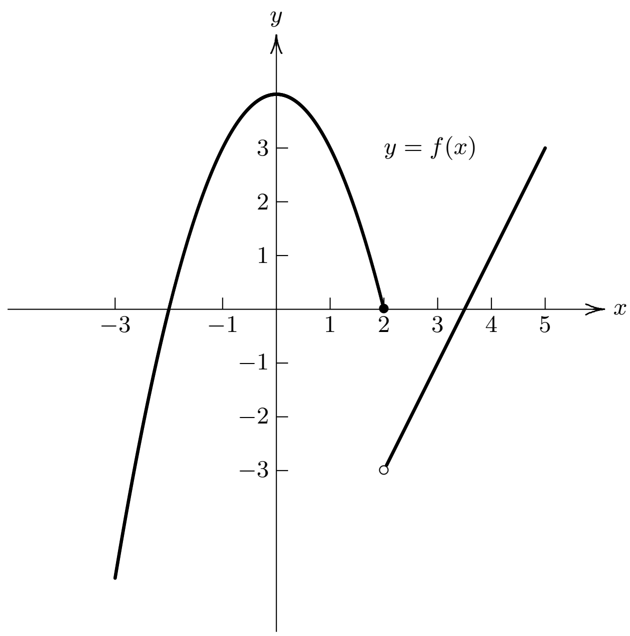
Solution: Using the notion of absolute value, we can write the inequality as a double inequality and then solve for x.

$$|3x - 1| < 7 \quad \Leftrightarrow \quad -7 < 3x - 1 < 7 \quad \Leftrightarrow \quad -6 < 3x < 8 \quad \Leftrightarrow \quad -2 < x < \frac{8}{3}.$$

It follows that the set $\{x \in \mathbb{R} : 4 + |3x - 1| < 11\}$ can be represented as the interval $(-2, \frac{8}{3})$.

Example 5: Sketch a graph of the function f defined by $f(x) = \begin{cases} 4 - x^2, & \text{if } x \le 2; \\ 2x - 7, & \text{if } x > 2. \end{cases}$

Solution: Since we know that the graph is a parabola for $x \le 2$ and a line for $x > 2$, we can easily sketch each part. The bullet versus the open circle indicates that the value of the function at 2 is 0.



Of course, the graph of the function continues indefinitely in each direction. ■

Example 6: For the function $f(x) = x - 4x^2$, find and simplify the quantity $\dfrac{f(x+h) - f(x-h)}{2h}$.

Solution: Using composition of functions and being very careful with the algebra (especially with regard to minus signs and parentheses), we find that

$$\begin{aligned}
\frac{f(x+h) - f(x-h)}{2h} &= \frac{\Big((x+h) - 4(x+h)^2\Big) - \Big((x-h) - 4(x-h)^2\Big)}{2h} \\
&= \frac{\Big(x + h - 4x^2 - 8hx - 4h^2\Big) - \Big(x - h - 4x^2 + 8hx - 4h^2\Big)}{2h} \\
&= \frac{2h - 16hx}{2h} \\
&= 1 - 8x.
\end{aligned}$$

We will see the geometric significance of these sorts of fractions later in the chapter. ■

Extra Notes for Section 1.3

Example 1: Evaluate $\lim_{x\to 2} \dfrac{x^3-8}{2x^2-7x+6}$.

Solution: Since the value of the fraction when $x = 2$ is the meaningless expression $0/0$, we know that $x - 2$ is a factor of each polynomial in the rational function. Factoring (and referring to the appendix if necessary to remember how to factor $a^3 - b^3$) yields

$$\lim_{x\to 2} \frac{x^3-8}{2x^2-7x+6} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(2x-3)(x-2)} = \lim_{x\to 2} \frac{x^2+2x+4}{2x-3} = 12.$$

The value of the limit is thus 12. As you do these problems, remember to write the limit sign until you actually evaluate the limit. ■

Example 2: Use a table of values to estimate $\lim_{x\to 1} \dfrac{\log_2 x}{x-1}$.

Solution: Letting f be the function defined by $f(x) = \log_2 x/(x-1)$, we find that

$$\begin{aligned} f(1.1) &= 1.375035; & f(0.9) &= 1.520031;\\ f(1.01) &= 1.435529; & f(0.99) &= 1.449957;\\ f(1.001) &= 1.441974; & f(0.999) &= 1.443417;\\ f(1.0001) &= 1.442623; & f(0.9999) &= 1.442767;\\ f(1.00001) &= 1.442688; & f(0.99999) &= 1.442702. \end{aligned}$$

(Ask someone for help if you do not know how to evaluate logarithms with different bases on your calculator.) From this table of values, it appears that $\lim_{x\to 1} \dfrac{\log_2 x}{x-1} \approx 1.443$. (The squiggly equals sign indicates that the proposed value for the limit is only an approximation.) ■

Example 3: Use properties of the greatest integer function to evaluate $\lim_{x\to 5^+} \big(x - \lfloor -x \rfloor\big)$.

Solution: Since the requested limit is a right-hand limit, we are interested in the values of the function $f(x) = x - \lfloor -x \rfloor$ when x is just a little bigger than 5. For example, we find that

$$\begin{aligned} f(5.4) &= 5.4 - \lfloor -5.4 \rfloor = 5.4 - (-6) = 11.4,\\ f(5.1) &= 5.1 - \lfloor -5.1 \rfloor = 5.1 - (-6) = 11.1,\\ f(5.02) &= 5.02 - \lfloor -5.02 \rfloor = 5.02 - (-6) = 11.02. \end{aligned}$$

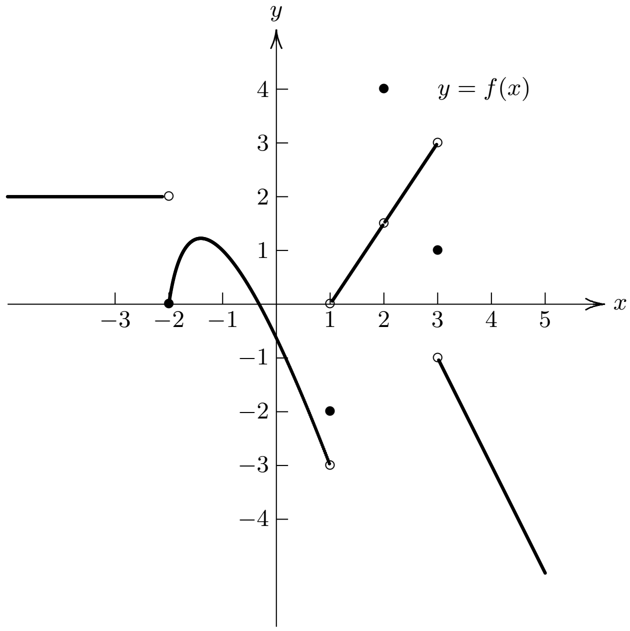
In general, when x is just a little bigger than 5, the number $-x$ is just a little smaller than -5. This means that $\lfloor -x \rfloor$ has the value -6. It follows that

$$\lim_{x\to 5^+} \big(x - \lfloor -x \rfloor\big) = 5 - (-6) = 11.$$

Therefore, the value of the limit is 11. ■

Example 4: Referring to the graph, find the following limits:

$$\lim_{x\to -2^-} f(x),\ \lim_{x\to -2^+} f(x),\ \lim_{x\to 1^-} f(x),\ \lim_{x\to 1^+} f(x),\ \lim_{x\to 2} f(x),\ \lim_{x\to 3^-} f(x),\ \lim_{x\to 3^+} f(x),\ \lim_{x\to 4} f(x).$$



Solution: Using the concept of one-sided limits and looking carefully at the graph to determine the values of the function $f(x)$, we find that

$$\begin{aligned} &\lim_{x\to -2^-} f(x) = 2; && \lim_{x\to 2} f(x) = \tfrac{3}{2}; \\ &\lim_{x\to -2^+} f(x) = 0; && \lim_{x\to 3^-} f(x) = 3; \\ &\lim_{x\to 1^-} f(x) = -3; && \lim_{x\to 3^+} f(x) = -1; \\ &\lim_{x\to 1^+} f(x) = 0; && \lim_{x\to 4} f(x) = -3. \end{aligned}$$

(Use simple values or fractions for the limits in these sorts of problems when the values are not 'obvious' from the graph.) Remember that the value of the function at the point c does not play a role in the computation of the limit at c. ■

Extra Notes for Section 1.4

Example 1: Find all of the discontinuities of the function f defined by $f(x) = \dfrac{x+5}{x^4 - 5x^2 + 6}$.

Solution: Since f is a rational function, it is continuous at all values for which it is defined. The only potential problem is when the denominator is equal to 0. Factoring the denominator, we find that

$$x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3).$$

It follows that the discontinuities of f are the four points $-\sqrt{2}$, $\sqrt{2}$, $-\sqrt{3}$, and $\sqrt{3}$. ■

Example 2: Prove that the polynomial $x^7 + 6x^2 - 40$ has a positive root.

Solution: Define a function P by $P(x) = x^7 + 6x^2 - 40$ for all real numbers x. Recall that a root of the polynomial P is a number r for which $P(r) = 0$. By Theorem 1.3, the function P is continuous for all real numbers. Simple calculation reveals that

$$P(1) = 1 + 6 - 40 = -33 \qquad \text{and} \qquad P(2) = 128 + 24 - 40 = 112.$$

Since 0 is between $P(1)$ and $P(2)$, the Intermediate Value Theorem guarantees the existence of a number $c \in (1,2)$ for which $P(c) = 0$. It follows that P has a positive root. (Note that your calculator cannot find the exact value of this root; it can only give you an approximation.) ■

Example 3: Prove that the equation $\cos x = 4x$ has a solution.

Solution: Define a function f by $f(x) = \cos x - 4x$. The function f is continuous on $\mathbb{R}$ by Theorem 1.3. In addition, we find that

$$f(0) = 1 \qquad \text{and} \qquad f(\pi/2) = -2\pi.$$

By the Intermediate Value Theorem, there exists a point $c \in (0, \pi/2)$ for which $f(c) = 0$. This means that $\cos c - 4c = 0$. It follows that c is a solution to the equation $\cos x = 4x$. ■

Example 4: Suppose that the functions f and g are continuous at c. Prove that the function $f - g$ is continuous at c.

Solution: Since f and g are continuous at c, we know that $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. By property (3) of limits given in Section 1.3, we then have

$$\lim_{x \to c} \big(f(x) - g(x)\big) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = f(c) - g(c).$$

This shows that the function $f - g$ is continuous at c. ■

Example 5: Find a value for the constant a so that the function f defined by $f(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x < 1; \\ ax + 2, & \text{if } x \geq 1. \end{cases}$ is continuous at 1.

Solution: In order for f to be continuous at 1, we need $\lim_{x\to 1} f(x) = f(1)$. For the limit to exist, the two one-sided limits must exist and be equal. Using the definition of the function f, we find that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \lfloor x \rfloor = 0 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax + 2) = a + 2.$$

We thus need $a = -2$. For this value of a, we have $\lim_{x\to 1} f(x) = 0 = f(1)$, revealing that the function f is continuous at 1. ■

Example 6: Find a negative number a so that the function f defined by $f(x) = \begin{cases} x^2, & \text{if } x \leq a; \\ x + 12, & \text{if } x > a. \end{cases}$ is continuous at a.

Solution: In order for f to be continuous at a, we need $\lim_{x\to a} f(x) = f(a)$. For the limit to exist, the two one-sided limits must exist and be equal. Using the definition of the function f, we find that

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x^2 = a^2 \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = \lim_{x\to a^+} (x + 12) = a + 12.$$

The number a must satisfy the equation $a^2 = a + 12$. Setting the equation equal to 0 and factoring yields

$$a^2 - a - 12 = 0 \quad \Leftrightarrow \quad (a - 4)(a + 3) = 0.$$

Since a must be negative, the desired value for a is -3. ■

Extra Notes for Section 1.5

Example 1: Use algebra to evaluate $\lim_{x\to 4} \frac{x^2+3x-28}{x^2-4x}$.

Solution: Since both the numerator and denominator equal 0 when $x = 4$, we know that $x - 4$ is a factor of both polynomials. It follows that

$$\begin{aligned}\lim_{x\to 4} \frac{x^2+3x-28}{x^2-4x} &= \lim_{x\to 4} \frac{(x-4)(x+7)}{x(x-4)} \\ &= \lim_{x\to 4} \frac{x+7}{x} \\ &= \frac{11}{4}.\end{aligned}$$

At the last step, we are using the continuity of the function $(x+7)/x$ at 4 to find the limit; for continuous functions, the value of the limit at c is the value of the function at c. ■

Example 2: Use algebra to evaluate $\lim_{x\to -3} \frac{\frac{1}{2x+7}-1}{x^2-2x-15}$.

Solution: For this limit, we first find a common denominator so we can add the fractions, then look for common factors. We find that

$$\begin{aligned}\lim_{x\to -3} \frac{\frac{1}{2x+7}-1}{x^2-2x-15} &= \lim_{x\to -3} \frac{\frac{1-(2x+7)}{2x+7}}{x^2-2x-15} \\ &= \lim_{x\to -3} \frac{\frac{-2x-6}{2x+7}}{(x+3)(x-5)} \\ &= \lim_{x\to -3} \frac{-2(x+3)}{(2x+7)(x+3)(x-5)} \\ &= \lim_{x\to -3} \frac{-2}{(2x+7)(x-5)} \\ &= \frac{-2}{(1)(-8)} = \frac{1}{4}.\end{aligned}$$

If you are sketchy on your algebra skills, study each step carefully. ■

Example 3: Use limits determined in the section to evaluate $\lim_{x\to 0} \frac{\sin x}{\sin(\pi x)}$.

Solution: For this limit, we simply rewrite the expression and use the fact that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

$$\lim_{x\to 0} \frac{\sin x}{\sin(\pi x)} = \lim_{x\to 0} \left(\frac{1}{\pi} \cdot \frac{\sin x}{x} \cdot \frac{\pi x}{\sin(\pi x)}\right) = \frac{1}{\pi} \cdot 1 \cdot 1 = \frac{1}{\pi}.$$

All we did was multiply by 1, but we did so in a convenient way to make the known limits appear. ■

Example 4: Evaluate $\lim_{x\to 0} \dfrac{1-\cos^2 x}{2x^2}$.

Solution: Recognizing a basic trigonometric identity, we find that

$$\lim_{x\to 0} \frac{1-\cos^2 x}{2x^2} = \lim_{x\to 0} \frac{\sin^2 x}{2x^2} = \lim_{x\to 0} \frac{1}{2}\Big(\frac{\sin x}{x}\Big)^2 = \frac{1}{2}.$$

We have once again used the fact that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$. ■

Example 5: Use algebra to evaluate $\lim_{x\to 5} \dfrac{\sqrt{x^2+3x+9}-7}{x-5}$.

Solution: This time, we multiply by the conjugate of the expression in the numerator.

$$\begin{aligned}
\lim_{x\to 5} \frac{\sqrt{x^2+3x+9}-7}{x-5} &= \lim_{x\to 5}\Big(\frac{\sqrt{x^2+3x+9}-7}{x-5}\cdot\frac{\sqrt{x^2+3x+9}+7}{\sqrt{x^2+3x+9}+7}\Big) \\
&= \lim_{x\to 5} \frac{x^2+3x-40}{(x-5)\big(\sqrt{x^2+3x+9}+7\big)} \\
&= \lim_{x\to 5} \frac{(x-5)(x+8)}{(x-5)\big(\sqrt{x^2+3x+9}+7\big)} \\
&= \lim_{x\to 5} \frac{x+8}{\sqrt{x^2+3x+9}+7} \\
&= \frac{13}{\sqrt{49}+7} = \frac{13}{14}.
\end{aligned}$$

Once again, pay careful attention to the algebraic details as well as the correct use of notation. ■

Example 6: Use the squeeze theorem to evaluate $\lim_{x\to 0^+} \big(2^{1/x}+3^{1/x}\big)^x$.

Solution: Let $f(x) = \big(2^{1/x}+3^{1/x}\big)^x$. For each value of x that satisfies $0 < x < 1$, we find that

$$1 < \frac{1}{x} < \infty \qquad \text{and thus} \qquad 2^{1/x} < 3^{1/x}.$$

Using this fact, we find that

$$3^{1/x} < 2^{1/x} + 3^{1/x} < 3^{1/x} + 3^{1/x} \qquad \text{and hence} \qquad 3 < f(x) < 3\cdot 2^x.$$

(Think carefully about these steps and write out some details if you are confused.) Since $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} 3\cdot 2^x = 3$, it follows from the squeeze theorem that $\lim_{x\to 0^+} f(x) = 3$. We conclude that

$$\lim_{x\to 0^+} \big(2^{1/x}+3^{1/x}\big)^x = 3.$$

As illustrated with this problem, it is sometimes challenging to find the two functions with the same limit needed to squeeze the given function. The basic idea in this case is that for small positive values of x, the number $3^{1/x}$ is much larger than $2^{1/x}$. It is so much larger that the term $2^{1/x}$ essentially disappears and we are left with a number that resembles $\big(3^{1/x}\big)^x = 3$. ■

Extra Notes for Section 1.6

Example 1: Find the slope of the curve $y = x^2 - 4x$ at $x = 3$.

Solution: Using Definition 1.6, we find that

$$\left.\frac{dy}{dx}\right|_{x=3} = \lim_{v \to 3} \frac{(v^2 - 4v) - (-3)}{v - 3} = \lim_{v \to 3} \frac{v^2 - 4v + 3}{v - 3} = \lim_{v \to 3} \frac{(v-3)(v-1)}{v - 3} = \lim_{v \to 3}(v - 1) = 2.$$

Alternatively, we can let $f(x) = x^2 - 4x$ and compute the following limit:

$$\begin{aligned}\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{(3+h)^2 - 4(3+h) - (-3)}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 12 - 4h + 3}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0}(2 + h) = 2.\end{aligned}$$

With either method, we find that the slope of the curve $y = x^2 - 4x$ at $x = 3$ is 2. ■

Example 2: Find an equation for the line tangent to the curve $y = x^2 - 4x$ when $x = 3$.

Solution: When $x = 3$, the value of y is -3 so the point on the graph is $(3, -3)$. Referring to the previous example, the slope of the curve at this point is 2. An equation for the tangent line is thus

$$y - (-3) = 2(x - 3) \qquad \text{or} \qquad y = 2x - 9.$$

It is usually easier (as we have done here) to use the point-slope form for the equation of a line for these types of problems. ■

Example 3: Find the slope of the graph of $f(x) = 4/(x + 1)$ at a generic point $c \neq -1$. Use your answer to find an equation for the normal line to the graph at $x = 3$.

Solution: Using Definition 1.6, we find that the slope of the graph at c is

$$\begin{aligned}\lim_{v \to c} \frac{\dfrac{4}{v+1} - \dfrac{4}{c+1}}{v - c} &= \lim_{v \to c} \frac{4}{v - c}\left(\frac{(c+1) - (v+1)}{(v+1)(c+1)}\right) \\ &= \lim_{v \to c} \frac{4(c - v)}{(v - c)(v+1)(c+1)} \\ &= \lim_{v \to c} \frac{-4}{(v+1)(c+1)} \\ &= -\frac{4}{(c+1)^2}.\end{aligned}$$

(Be certain you understand each of the algebra steps.) The point $c = -1$ is omitted because it is not in the domain of the function. In particular, the slope of the line tangent to the graph when $c = 3$ is $-1/4$. Since the normal line is perpendicular to the tangent line, it must have a slope of 4. It follows that an equation for the normal line to the graph at $x = 3$ is $y - 1 = 4(x - 3)$ or $y = 4x - 11$. ■

Example 4: Let ℓ_1 be the tangent line to the graph of $y = x^3$ at $x = -1$ and let ℓ_2 be the normal line to the graph of $y = x^3$ at $x = 1$. Find the point of intersection of ℓ_1 and ℓ_2.

Solution: Referring to Exercise 4 in Section 1.6, we know that the slope of the curve $y = x^3$ at a point c is $3c^2$. The equations for the lines ℓ_1 and ℓ_2 are thus

$$y - (-1) = 3(x - (-1)) \qquad \text{and} \qquad y - 1 = -\frac{1}{3}(x - 1),$$

respectively. Solving each equation for y, setting them equal to each other, and then solving for x yields

$$3(x+1) - 1 = -\frac{1}{3}(x-1) + 1 \quad \Leftrightarrow \quad 3x + 2 = -\frac{1}{3}x + \frac{4}{3} \quad \Leftrightarrow \quad \frac{10}{3}x = -\frac{2}{3} \quad \Leftrightarrow \quad x = -\frac{1}{5}.$$

Once we know x, we can use either equation to find y:

$$y = 3\Big(-\frac{1}{5}\Big) + 2 = \frac{7}{5}.$$

The point of intersection of the lines ℓ_1 and ℓ_2 is $\big(-\frac{1}{5}, \frac{7}{5}\big)$. ■

Example 5: Let f be the function defined by $f(x) = x^2$. Find the (x, y) coordinates of a point on the graph of $y = f(x)$ at which the tangent line to the graph goes through the point $(4, 7)$.

Solution: Let c be a generic real number. By Exercise 2 in Section 1.6, the slope of the curve $y = x^2$ at c is $2c$. Hence, an equation for the tangent line at this point is $y - c^2 = 2c(x - c)$. We want to find a value of c for which this line contains the point $(4, 7)$. Substituting $x = 4$ and $y = 7$ into the equation and then solving for c yields

$$7 - c^2 = 2c(4 - c) \quad \Leftrightarrow \quad c^2 - 8c + 7 = 0 \quad \Leftrightarrow \quad (c-1)(c-7) = 0.$$

Therefore, at each of the points $(1, 1)$ and $(7, 49)$, the tangent line to the graph of $f(x) = x^2$ goes through the point $(4, 7)$. ■

Example 6: The position $p(t)$ of a particle (in meters) at time t (in seconds) is given by $p(t) = t^2 - 6t + 10$. Determine the velocity of the particle after four seconds.

Solution: Since the slope of a distance function gives velocity, we can use Definition 1.6 to find velocity. In this case, we have

$$\lim_{t \to 4} \frac{p(t) - p(4)}{t - 4} = \lim_{t \to 4} \frac{(t^2 - 6t + 10) - 2}{t - 4} = \lim_{t \to 4} \frac{(t-2)(t-4)}{t - 4} = \lim_{t \to 4}(t - 2) = 2.$$

The velocity of the particle after four seconds is two meters per second. (Remember to provide units for your answer when they are given in the problem.) ■

Extra Notes for Section 1.7

Example 1: Find the derivative of the function f defined by $f(x) = 2x^2 + 3x - 5$.

Solution: Using the $v \to x$ version of the definition, we obtain

$$\begin{aligned} f'(x) &= \lim_{v\to x} \frac{(2v^2+3v-5)-(2x^2+3x-5)}{v-x} \\ &= \lim_{v\to x} \frac{2(v^2-x^2)+3(v-x)}{v-x} \\ &= \lim_{v\to x} \big(2(v+x)+3\big) \\ &= 4x+3. \end{aligned}$$

For the sake of comparison, the $h \to 0$ version yields

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\big(2(x+h)^2+3(x+h)-5\big)-(2x^2+3x-5)}{h} \\ &= \lim_{h\to 0} \frac{2x^2+4xh+2h^2+3x+3h-5-2x^2-3x+5}{h} \\ &= \lim_{h\to 0} \frac{4xh+2h^2+3h}{h} \\ &= \lim_{h\to 0} (4x+2h+3) \\ &= 4x+3. \end{aligned}$$

The first version of the definition typically involves more factoring while the second version yields more terms when quantities are expanded. ■

Example 2: Find all of the values of x for which $f'(x) = 3/2$ given that $f(x) = x - (4/x)$.

Solution: We begin by finding $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{v\to x} \frac{\left(v-\frac{4}{v}\right)-\left(x-\frac{4}{x}\right)}{v-x} \\ &= \lim_{v\to x} \frac{v-x+4\left(\frac{1}{x}-\frac{1}{v}\right)}{v-x} \\ &= \lim_{v\to x} \frac{v-x+4\left(\frac{v-x}{xv}\right)}{v-x} \\ &= \lim_{v\to x} \left(1+\frac{4}{xv}\right) \\ &= 1+\frac{4}{x^2}. \end{aligned}$$

Solving the equation $f'(x) = 3/2$, we find that

$$1+\frac{4}{x^2}=\frac{3}{2} \quad \Leftrightarrow \quad \frac{4}{x^2}=\frac{1}{2} \quad \Leftrightarrow \quad x^2=8.$$

The equation $f'(x) = 3/2$ is thus satisfied when $x = \pm 2\sqrt{2}$. ■

Example 3: Find the x-intercept of the normal line to the curve $y = x^3 + 4x^2$ when $x = 2$.

Solution: Using the definition to find the derivative, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \lim_{v\to x} \frac{(v^3+4v^2)-(x^3+4x^2)}{v-x} \\ &= \lim_{v\to x} \frac{(v^3-x^3)+4(v^2-x^2)}{v-x} \\ &= \lim_{v\to x} \big((v^2+vx+x^2)+4(v+x)\big) \\ &= 3x^2+8x.\end{aligned}$$

Note that we have (once again) used the factoring formula for $a^3 - b^3$. When $x = 2$, we have

$$\left.\frac{dy}{dx}\right|_{x=2} = 3 \cdot 2^2 + 8 \cdot 2 = 28.$$

(Do pay attention to the notation in this case; there is a distinction between the general derivative and the derivative at a particular point.) The slope of the tangent line is 28 so the slope of the normal line is $-1/28$. When $x = 2$, it is easy to see that $y = 24$. It follows that an equation for the normal line is

$$y - 24 = -\frac{1}{28}(x-2)$$

The x-intercept of this line (which occurs when $y = 0$) is $x = 24 \cdot 28 + 2 = 674$. ■

Example 4: Find an equation for a function f that is continuous for all real numbers and differentiable for all real numbers except -1, 0, and 6.

Solution: We know that $|x|$ is continuous at 0 but not differentiable at 0. Using this idea, we see that the function f defined by

$$f(x) = |x+1| + |x| + |x-6|$$

has the desired properties. (What theorem in the next section is implicitly being used here?) A graph (not to scale) of this function is given below; note the corners at the points of nondifferentiability.