Extra Notes for Section 2.1

Example 1: Without using a calculator, evaluate $21^3 + 22^3 + 23^3 + \cdots + 40^3$.

Solution: The expression $21^3 + 22^3 + 23^3 + \cdots + 40^3$ can be written as $\sum_{i=21}^{40} i^3$. To find the value of this sum, we need to rewrite the sum so that the first index is $i = 1$, then use the formula for the sum of cubes. Doing so yields

$$\begin{aligned}\sum_{i=21}^{40} i^3 &= \sum_{i=1}^{40} i^3 - \sum_{i=1}^{20} i^3 \\ &= \Big(\frac{40 \cdot 41}{2}\Big)^2 - \Big(\frac{20 \cdot 21}{2}\Big)^2 \\ &= 10^2\big(82^2 - 21^2\big) \\ &= 100 \cdot 61 \cdot 103 = 100(6100 + 183) = 628300.\end{aligned}$$

Although you may think a calculator is more appropriate for this problem, see if you can follow the reasoning behind each step. Some of the ideas that appear are useful. If you imagine the work involved with finding the twenty cubes and then adding them up, all without a calculator, you can appreciate how helpful the sum formula is.

Example 2: Find a simple formula for the sum $6 + 11 + 16 + 21 + \cdots + (5n + 11)$.

Solution: Referring to the last term of the sum, it appears that the terms in the sum are of the form $5i + c$ for some constant c. In order for the first term ($i = 1$) to be 6, we need $c = 1$. This means the terms are of the form $5i + 1$. The last term then corresponds to $i = n + 2$ since $5(n + 2) + 1 = 5n + 11$. (If you do not see this, simply solve the equation $5i + 1 = 5n + 11$ for i.) Using formulas for sums, we find that

$$\begin{aligned}6 + 11 + 16 + 21 + \cdots + (5n + 11) &= \sum_{i=1}^{n+2}(5i + 1) = 5\sum_{i=1}^{n+2} i + \sum_{i=1}^{n+2} 1 \\ &= 5 \cdot \frac{(n + 2)(n + 3)}{2} + (n + 2) \\ &= \frac{n + 2}{2}\big(5(n + 3) + 2\big) = \frac{1}{2}\,(n + 2)(5n + 17).\end{aligned}$$

Alternatively, since the sum is arithmetic (the difference between consecutive terms is constant), we can use Gauss' idea and write

$$\begin{aligned}S &= \quad 6 \quad + \quad 11 \quad + \quad 16 \quad + \cdots + (5n + 6) + (5n + 11) \\ S &= (5n + 11) + (5n + 6) + (5n + 1) + \cdots + \quad 11 \quad + \quad 6\end{aligned}$$

Adding the two expressions for S AND realizing that there are $n + 2$ terms in the sum (using ideas from above) gives us $2S = (n + 2)(5n + 17)$, the same result we found with the other method. This method may seem faster but you must be careful determining the number of terms. Also, note that this approach is only valid for arithmetic sums.

Example 3: Evaluate $\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=2n-1}^{3n} i^2$.

Solution: Before evaluating this limit, let's try to understand what is going on here. For the first few values of n, we find that

$$n = 1 \text{ gives } \frac{1}{1} \sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14;$$

$$n = 2 \text{ gives } \frac{1}{2^3} \sum_{i=3}^{6} i^2 = \frac{9 + 16 + 25 + 36}{8} = \frac{43}{4};$$

$$n = 3 \text{ gives } \frac{1}{3^3} \sum_{i=5}^{9} i^2 = \frac{25 + 36 + 49 + 64 + 81}{27} = \frac{85}{9};$$

and, jumping ahead a bit,

$$n = 20 \text{ gives } \frac{1}{20^3} \sum_{i=39}^{60} i^2 = \frac{54791}{8000} = 6.848875.$$

We are asked to determine what happens to these numbers as n increases. Using our formulas for sums, we find that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=2n-1}^{3n} i^2 &= \lim_{n\to\infty} \frac{1}{n^3} \Big(\sum_{i=1}^{3n} i^2 - \sum_{i=1}^{2n-2} i^2 \Big) \\
&= \lim_{n\to\infty} \frac{1}{n^3} \Big(\frac{3n(3n+1)(6n+1)}{6} - \frac{(2n-2)(2n-1)(4n-3)}{6} \Big) \\
&= \lim_{n\to\infty} \Big(\frac{3n(3n+1)(6n+1)}{6n^3} - \frac{(2n-2)(2n-1)(4n-3)}{6n^3} \Big) \\
&= 9 - \frac{8}{3} = \frac{19}{3}.
\end{aligned}$$

There are several observations worth noting here. It is important to realize that the value of n used in the sum formulas represents a generic number, the stopping point of the sum. We could write the formula for the sum of the squares as

$$\sum_{i=1}^{\text{stop}} i^2 = \frac{\text{stop}(\text{stop} + 1)(2 \cdot \text{stop} + 1)}{6}.$$

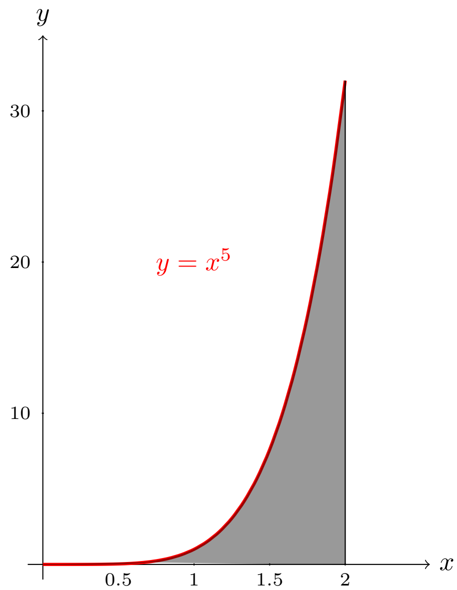
This is a rather cumbersome way to write a mathematical expression but it gets the basic idea across. For the sums in the equation, we used $3n$ and $2n - 2$ for the stop value. Be certain you understand this. The limits as $n \to \infty$ are of the form polynomial over polynomial. You should recall (or at least learn now) that when the degrees of the polynomials are the same, the value of the limit is simply the ratio of the leading coefficients of the polynomials. There is no need to multiply out the polynomials or to use L'Hôpital's Rule.

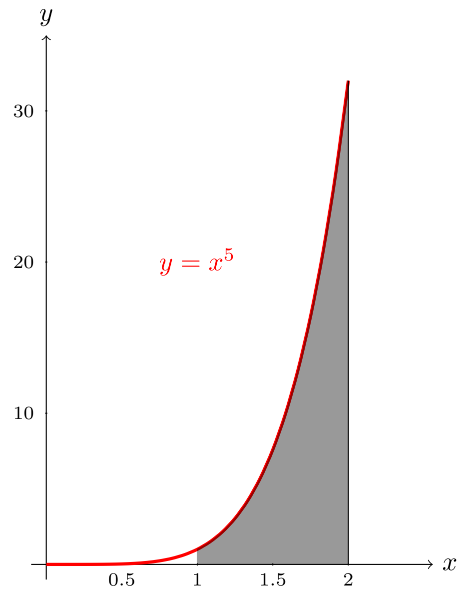
Example 4: Extend the table in Section 2.1 by adding a column for $\sum_{i=1}^{n} i^5$. Write each sum as a product of the number in the $\sum_{i=1}^{n} i^3$ column times a fraction with a denominator of 3. As in the text, find a formula for the numerator and thus obtain a formula for $\sum_{i=1}^{n} i^5$. Finding this formula is very similar to the example in the notes and is good practice. Show that $\sum_{i=1}^{n} i^5 = \frac{n^6}{6} + \frac{n^5}{2} +$ smaller positive integer powers of n.

Extra Notes for Section 2.2

Example 1: Find the area under the curve $y = x^5$ and above the x-axis on the intervals $[0, 2]$ and $[1, 2]$.

Solution: Let $A_{[a,b]}$ denote the area under the curve $y = x^5$ and above the x-axis on the interval $[a, b]$. We want to find $A_{[0,2]}$ and $A_{[1,2]}$; these areas are shaded below.



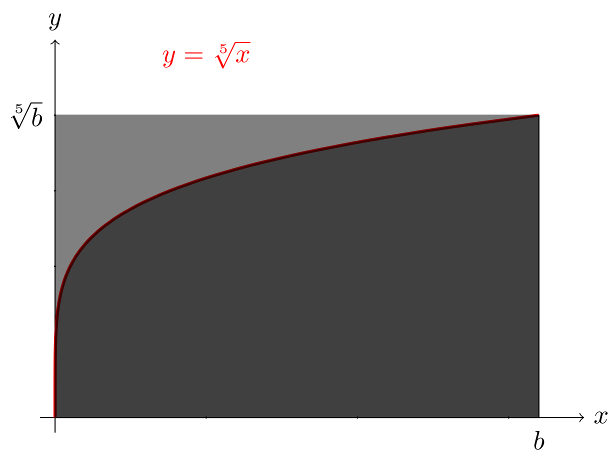


Using the formula derived in this section and some simple properties of area, we find that

$$A_{[0,2]} = \frac{2^6}{6} = \frac{32}{3} \quad \text{and} \quad A_{[1,2]} = A_{[0,2]} - A_{[0,1]} = \frac{2^6}{6} - \frac{1^6}{6} = \frac{63}{6} = \frac{21}{2}.$$

Example 2: Find the area under the curve $y = \sqrt[5]{x}$ and above the x-axis on the interval $[0, b]$.

Solution: Referring to the graph below, we want to find the area of the darkly shaded region.



If we use our area definition to write this value as a limit of sums, we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[5]{\frac{ib}{n}} \cdot \frac{b}{n} = \lim_{n\to\infty} \left(\frac{b}{n}\right)^{6/5} \sum_{i=1}^{n} \sqrt[5]{i}.$$

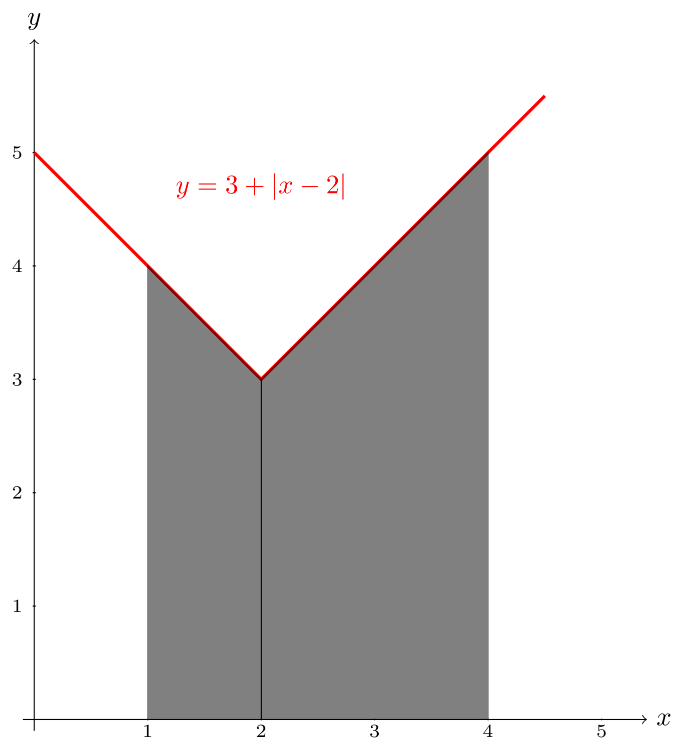
We have not developed a formula for the sum of the fifth roots of integers so we appear to be stuck. However, the graph of $y = \sqrt[5]{x}$ is the same as the graph of $x = y^5$. We know how to find the area under this curve if we reverse the axes. This might seem odd, but our choice of labels and directions for the axes is a habit, not a requirement. Using our formula for the area under a fifth power, we know that the area of the lightly shaded region is $(\sqrt[5]{b})^6/6 = b^{6/5}/6$. The area that we want, the area of the darkly shaded region, is the area of the entire shaded rectangle minus the area of the lightly shaded region. We thus find that the area under the curve $y = \sqrt[5]{x}$ and above the x-axis on the interval $[0, b]$ is

$$b \cdot \sqrt[5]{b} - \frac{1}{6}\, b^{6/5} = \frac{5}{6}\, b^{6/5} = \frac{b^{6/5}}{6/5}.$$

Note that this formula for the area fits the same pattern as the area formulas when r is a positive integer.

Example 3: Find the area under the curve $y = 3 + |x - 2|$ and above the x-axis on the interval $[1, 4]$.

Solution: The area under consideration is the shaded region sketched below.



In this case, we can use simple geometry. The region consists of two trapezoids, with vertical lines as the parallel sides. Recall that the area of a trapezoid with bases b and B and height h is given by $\frac{1}{2}\,(b + B)h$. The area A of the region is thus

$$A = \frac{1}{2}(4+3)\cdot 1 + \frac{1}{2}(3+5)\cdot 2 = \frac{7}{2} + 8 = \frac{23}{2}.$$

Extra Notes for Section 2.3

Example 1: Use the definition of the integral to evaluate $\int_2^3 (2x^2 - x)\,dx$.

Solution: Applying the definition of the integral to the function $f(x) = 2x^2 - x$ on the interval $[2,3]$ and using the formulas for sums from Section 2.1, we obtain

$$\begin{aligned}
\int_2^3 (2x^2 - x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\Big(2 + \frac{i}{n}\Big)^2 - \Big(2 + \frac{i}{n}\Big) \right) \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Big(8 + \frac{8i}{n} + \frac{2i^2}{n^2} - 2 - \frac{i}{n} \Big) \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Big(6 + \frac{7i}{n} + \frac{2i^2}{n^2} \Big) \frac{1}{n} \\
&= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} 6 + \frac{7}{n^2} \sum_{i=1}^{n} i + \frac{2}{n^3} \sum_{i=1}^{n} i^2 \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n} \cdot 6n + \frac{7}{n^2} \cdot \frac{n(n+1)}{2} + \frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right) \\
&= 6 + \frac{7}{2} + \frac{2}{3} = \frac{61}{6}.
\end{aligned}$$

As done here, you should be able to evaluate some limits of this type by inspection.

Example 2: Use the definition of the integral to evaluate $\int_0^1 2^x\,dx$.

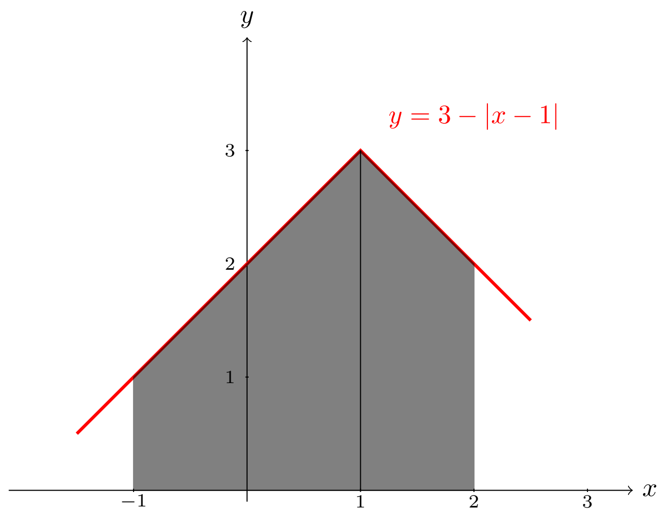
Solution: Applying the definition of the integral to the function $f(x) = 2^x$ on the interval $[0,1]$ and using the formula from problem 7 in Section 2.1 with $r = 2^{1/n}$, we obtain

$$\begin{aligned}
\int_0^1 2^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} 2^{i/n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \big(2^{1/n}\big)^i \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot 2^{1/n} \cdot \frac{\big(2^{1/n}\big)^n - 1}{2^{1/n} - 1} = \lim_{n\to\infty} 2^{1/n} \cdot \frac{1/n}{2^{1/n} - 1} \\
&= \lim_{n\to\infty} 2^{1/n} \cdot \lim_{n\to\infty} \frac{1/n}{2^{1/n} - 1} = 1 \cdot \lim_{x\to 0^+} \frac{x}{2^x - 1} \\
&= \lim_{x\to 0^+} \frac{1}{2^x \cdot \ln 2} = \frac{1}{\ln 2}.
\end{aligned}$$

There are several things to note here. First of all, you need to be careful with details and your use of notation. Secondly, you must realize that letters in formulas represent generic quantities; in this case we let $r = 2^{1/n}$ in a previous formula. Third, we let $x = 1/n$ in a step near the end and then noted that $x \to 0^+$ as $n \to \infty$. This step can be quite useful at times so you may want to remember it. Finally, we used L'Hôpital's Rule to evaluate the limit, using the derivative formula $\frac{d}{dx}\, a^x = a^x \cdot \ln a$.

Example 3: Evaluate $\displaystyle\int_{-1}^{2} \big(3 - |x-1|\big)\, dx$.

Solution: We begin by recalling that some integrals represent areas. Since the integrand in this case is positive on the given interval, the value of this integral represents the area under the curve $y = 3 - |x-1|$ and above the x-axis on the interval $[-1, 2]$. This region is sketched below.



The region in question represents two trapezoids and its area is $\dfrac{1+3}{2} \cdot 2 + \dfrac{3+2}{2} \cdot 1 = \dfrac{13}{2}$. It follows that $\displaystyle\int_{-1}^{2} \big(3 - |x-1|\big)\, dx = \frac{13}{2}$.

Example 4: Use the definition of the integral to write each limit as an integral.

Solution: We ask the reader to study each example carefully.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5\Big(4 + \frac{3i}{n}\Big)^3 + 2\Big(4 + \frac{3i}{n}\Big)\right) \cdot \frac{3}{n} = \int_4^7 (5x^3 + 2x)\, dx;$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5\Big(4 - \frac{3i}{n}\Big)^3 + 2\Big(4 - \frac{3i}{n}\Big)\right) \cdot \frac{3}{n} = \int_0^3 \big(5(4-x)^3 + 2(4-x)\big)\, dx;$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \Big(\frac{3i}{n}\Big)^2} \cdot \frac{1}{n} = \int_0^1 \sqrt{4 + 9x^2}\, dx;$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \Big(\frac{3i}{n}\Big)^2} \cdot \frac{1}{n} = \frac{1}{3} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \Big(\frac{3i}{n}\Big)^2} \cdot \frac{3}{n} = \frac{1}{3} \int_0^3 \sqrt{4 + x^2}\, dx;$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\Big(1 + \dfrac{2i}{n}\Big)^2} \cdot \frac{2}{n} = \int_1^3 \frac{1}{x^2}\, dx;$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1/n}{1 + (i/n)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1 + \Big(\dfrac{i}{n}\Big)^2} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x^2}\, dx.$$

Extra Notes for Section 2.4

Example 1: Evaluate the integral $\int_{-1}^{2} \left(2x^2 - \frac{x}{3} - 3\right) dx$.

Solution: Using the linearity properties of the integral and the formula for $\int_a^b x^r\,dx$ when r is a positive integer, we obtain

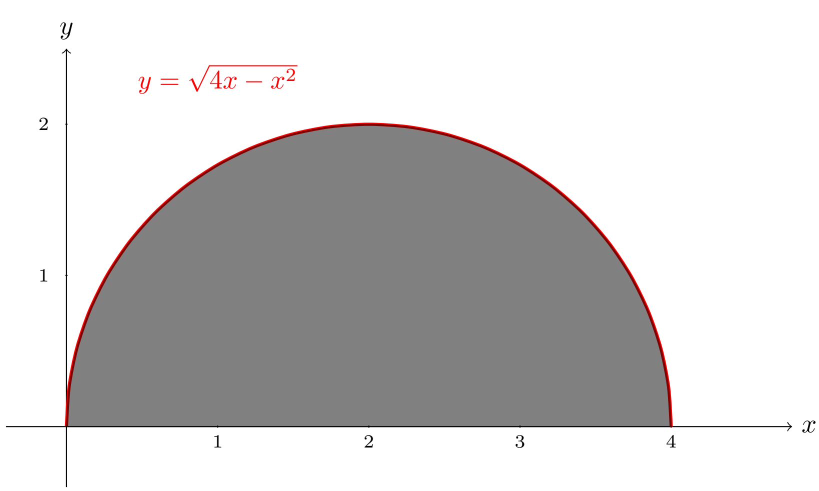
$$\begin{aligned}\int_{-1}^{2} \left(2x^2 - \frac{x}{3} - 3\right) dx &= 2\int_{-1}^{2} x^2\,dx - \frac{1}{3}\int_{-1}^{2} x\,dx - \int_{-1}^{2} 3\,dx \\ &= 2\cdot\frac{2^3-(-1)^3}{3} - \frac{1}{3}\cdot\frac{2^2-(-1)^2}{2} - 3\cdot 3 \\ &= 6 - \frac{1}{2} - 9 = -\frac{7}{2}.\end{aligned}$$

Example 2: Evaluate the integral $\int_0^4 \sqrt{4x - x^2}\,dx$.

Solution: Since the integrand is not a polynomial, we cannot use the integration formulas developed in this section. We clearly do not want to use the definition of the integral because the square root leads to very messy terms. What else can we do? Recall that some integrals represent areas. Since the integrand in this case is nonnegative, this integral represents the area under the curve $y = \sqrt{4x - x^2}$ and above the x-axis on the interval $[0, 4]$. If we can find this area using geometry, then we have found the value of the integral. Rewriting the equation, we find that

$$y = \sqrt{4x - x^2} \quad\Rightarrow\quad y^2 = 4x - x^2 \quad\Rightarrow\quad x^2 - 4x + 4 + y^2 = 4 \quad\Rightarrow\quad (x-2)^2 + y^2 = 4.$$

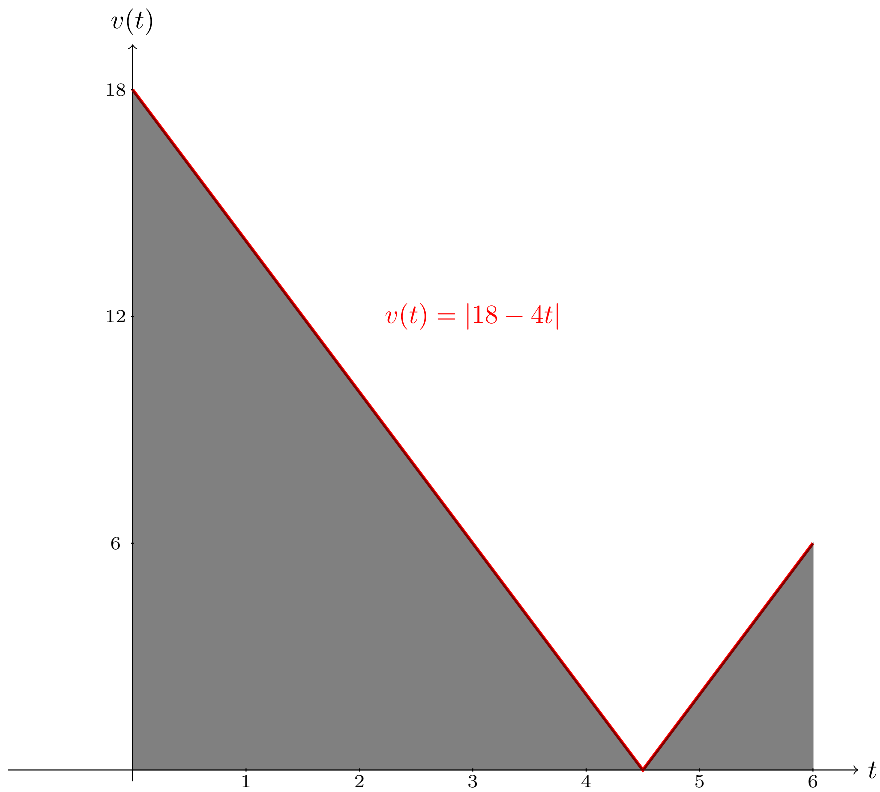
Note that we have used the technique of completing the square, something that you need to know. We now recognize that this function represents a portion of the circle of radius 2 centered at $(2, 0)$.



The area of the shaded region is $\frac{1}{2}(\pi \cdot 2^2) = 2\pi$ and we thus find that $\int_0^4 \sqrt{4x - x^2}\,dx = 2\pi$.

Example 3: Suppose that $v(t) = 18 - 4t$ represents the velocity in meters per second of a particle at time t seconds. Find the distance traveled by the particle for $0 \le t \le 6$.

Solution: By problem 6 in Section 2.3, we know that the distance traveled by the particle during this time period is $\int_0^6 |v(t)|\,dt = \int_0^6 |18 - 4t|\,dt$. Since the integrand involves the absolute value of a linear function, we can use geometry to evaluate this integral.



The integral represents the area of the two shaded triangles so

$$\int_0^6 |18 - 4t|\,dt = \frac{1}{2} \cdot \frac{9}{2} \cdot 18 + \frac{1}{2} \cdot \frac{3}{2} \cdot 6 = \frac{81}{2} + \frac{9}{2} = 45.$$

We conclude that the particle travels 45 meters during the time period $0 \le t \le 6$.

For a different method, we can recognize that the particle travels to the right (the positive direction) for $0 \le t \le 4.5$ and to the left (the negative direction) for $4.5 \le t \le 6$. The total distance traveled by the particle (in meters) is thus

$$\begin{aligned}\int_0^{4.5} (18 - 4t)\,dt + \left(-\int_{4.5}^{6} (18 - 4t)\,dt \right) &= \int_0^{4.5} 18\,dt - 4\int_0^{4.5} t\,dt + 4\int_{4.5}^{6} t\,dt - \int_{4.5}^{6} 18\,dt \\ &= 18 \cdot \frac{9}{2} - 4 \cdot \frac{(9/2)^2 - 0^2}{2} + 4 \cdot \frac{6^2 - (9/2)^2}{2} - 18 \cdot \frac{3}{2} \\ &= 81 - \frac{81}{2} + \frac{63}{2} - 27 = 45.\end{aligned}$$

Which method for evaluating this integral do you find is easier or makes more sense?

Extra Notes for Section 2.5

Example 1: Find upper and lower bounds for the value of $\int_1^4 \sqrt{30x - x^3}\, dx$.

Solution: We will solve this problem using property (5) of the integral. To do so, we need to find the maximum and minimum values of the function $f(x) = \sqrt{30x - x^3}$ on the interval $[1, 4]$. Since

$$f'(x) = \frac{30 - 3x^2}{2\sqrt{30x - x^3}},$$

we easily find that the critical points for f are $\pm\sqrt{10}$, with only the value $\sqrt{10}$ lying in the relevant interval. Evaluating f at the critical point in the interval and at the endpoints yields

$$f(1) = \sqrt{29} \approx 5.385, \quad f\big(\sqrt{10}\big) = \sqrt{20\sqrt{10}} \approx 7.953, \quad f(4) = \sqrt{56} \approx 7.483.$$

It follows that the maximum value of f on $[1, 4]$ is $\sqrt{20\sqrt{10}}$ and the minimum value of f on $[1, 4]$ is $\sqrt{29}$. By property (5) of integrals, we find that (note that the length of the interval is 3)

$$3\sqrt{29} \le \int_1^4 \sqrt{30x - x^3}\, dx \le 3\sqrt{20\sqrt{10}}.$$

Therefore, the value of the integral lies between 16.15 and 23.86. (For the record, the actual value of the integral, which can be found with the aid of a calculator, is approximately 21.89.)

Example 2: Find a good underestimate for the value of $\int_{-1}^3 \sqrt{x^8 + 3}\, dx$.

Solution: Since the integrand $\sqrt{x^8 + 3}$ is rather complicated, we want to replace it with a simpler function that is a little smaller and easy to integrate. Omitting the $+3$ in the square root of the integrand reveals that $x^4 = \sqrt{x^8} \le \sqrt{x^8 + 3}$ for all x in the interval $[-1, 3]$. In addition, the function x^4 is easy to integrate. By property (6) of integrals, it follows that

$$\int_{-1}^3 x^4\, dx \le \int_{-1}^3 \sqrt{x^8 + 3}\, dx.$$

Evaluating the integral on the left yields

$$\int_{-1}^3 x^4\, dx = \frac{3^5 - (-1)^5}{5} = \frac{243 + 1}{5} = \frac{244}{5}.$$

A good underestimate for the value of $\int_{-1}^3 \sqrt{x^8 + 3}\, dx$ is thus 48.8. (The actual value of this integral is approximately 52.35.)

Example 3: Evaluate $\displaystyle\int_0^3 |4x - x^3|\,dx$.

Solution: Although the integrand involves the absolute value function, we cannot use geometry to evaluate this integral since the area it represents does not have straight line boundaries. However, we can use property (7) of the integral. We begin by noting that $4x - x^3 = 0$ when $x = -2, 0, 2$ and that

$$|4x - x^3| = \begin{cases} 4x - x^3, & \text{if } 0 \le x \le 2; \\ x^3 - 4x, & \text{if } 2 \le x \le 3. \end{cases}$$

(We are using the fact that $|x| = x$ when x is positive and $|x| = -x$ when x is negative since the symbol $|x|$ represents a positive quantity.) By property (7), we find that

$$\begin{aligned}
\int_0^3 |4x - x^3|\,dx &= \int_0^2 |4x - x^3|\,dx + \int_2^3 |4x - x^3|\,dx \\
&= \int_0^2 (4x - x^3)\,dx + \int_2^3 (x^3 - 4x)\,dx \\
&= 4\int_0^2 x\,dx - \int_0^2 x^3\,dx + \int_2^3 x^3\,dx - 4\int_2^3 x\,dx \\
&= 4 \cdot \frac{2^2 - 0^2}{2} - \frac{2^4 - 0^4}{4} + \frac{3^4 - 2^4}{4} - 4 \cdot \frac{3^2 - 2^2}{2} \\
&= 8 - 4 + \frac{65}{4} - 10 \\
&= \frac{65}{4} - 6 = \frac{41}{4}.
\end{aligned}$$

Example 4: Find the point c guaranteed by the Mean Value Theorem for integrals for the function f defined by $f(x) = x^3$ on the interval $[0, 4]$.

Solution: Applying the Mean Value Theorem for integrals to the function $f(x) = x^3$ on the interval $[0, 4]$, we see that there exists a point $c \in [0, 4]$ such that

$$4c^3 = \int_0^4 x^3\,dx = \frac{4^4 - 0^4}{4} = 4^3 = 64.$$

Solving this equation for c yields $c = \sqrt[3]{16}$.

Extra Notes for Section 2.6

Example 1: Find the derivative of the function f defined by $f(x) = \int_x^{x^2} \sin(2t^3)\,dt$.

Solution: Using a properties of integrals, we can express $f(x)$ as

$$f(x) = \int_x^0 \sin(2t^3)\,dt + \int_0^{x^2} \sin(2t^3)\,dt = -\int_0^x \sin(2t^3)\,dt + \int_0^{x^2} \sin(2t^3)\,dt.$$

By the Fundamental Theorem of Calculus, it then follows that

$$f'(x) = -\sin(2x^3) + \sin(2(x^2)^3)\cdot 2x = 2x\sin(2x^6) - \sin(2x^3).$$

Note the use of the Chain Rule when finding the derivative of the second integral.

Example 2: Evaluate $\lim_{x\to 0} \frac{1}{x^3}\int_0^{2x} \left(1 - e^{-t^2}\right) dt$.

Solution: Since the limit is in the indeterminate form $\infty \cdot 0$, we perform some algebra and then use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x^3}\int_0^{2x} \left(1 - e^{-t^2}\right) dt &= \lim_{x\to 0} \frac{\int_0^{2x} \left(1 - e^{-t^2}\right) dt}{x^3} && \frac{0}{0} \text{ form, L'Hôpital's Rule} \\
&= \lim_{x\to 0} \frac{2\left(1 - e^{-4x^2}\right)}{3x^2} && \text{use FTC for numerator} \\
&= \lim_{x\to 0} \frac{16xe^{-4x^2}}{6x} && \text{use L'Hôpital's Rule again} \\
&= \lim_{x\to 0} \frac{8e^{-4x^2}}{3} && \text{do some simple algebra} \\
&= \frac{8}{3}. && \text{compute the limit}
\end{aligned}$$

Example 3: Find a function f for which $f(\pi) = 8$ and $f'(x) = e^x\cos(x^2)$ for all x.

Solution: We cannot write down a function with this property in the traditional way. However, we can use the Fundamental Theorem of Calculus to express the requested function in integral form. Consider the function f defined by

$$f(x) = \int_\pi^x e^t\cos(t^2)\,dt + 8.$$

It is easy to see that $f'(x) = e^x\cos(x^2)$ (by the FTC) and that $f(\pi) = 0 + 8 = 8$ (since $\int_a^a g(t)\,dt = 0$). Hence, the function f has the desired properties.

Extra Notes for Section 2.7

In order to find antiderivatives, it is necessary to know how to find derivatives well. For this section, the goal is to remember the basic derivative formulas and think about them in reverse.

Example 1: Evaluate $\displaystyle\int_{-2}^{3} \big(2x^2 - 4x - 7\big)\,dx$.

Solution: It is easy to find antiderivatives of polynomials. Doing so in this case, we find that

$$\begin{aligned}\int_{-2}^{3} \big(2x^2 - 4x - 7\big)\,dx &= \Big(\frac{2}{3}\,x^3 - 2x^2 - 7x\Big)\Big|_{-2}^{3} \\ &= \big(18 - 18 - 21\big) - \Big(-\frac{16}{3} - 8 + 14\Big) \\ &= -21 + \frac{16}{3} - 6 \\ &= -27 + \frac{16}{3} = -\frac{81}{3} + \frac{16}{3} = -\frac{65}{3}.\end{aligned}$$

You can always check that your antiderivative is correct by taking the derivative. Also, take your time with the arithmetic and combine the numbers in the simplest way possible.

Example 2: Evaluate $\displaystyle\int_{0}^{1} \big(2\sqrt[3]{x} - \sqrt{x}\big)\,dx$.

Solution: Converting to fractional exponents, we find that

$$\int_{0}^{1} \big(2\sqrt[3]{x} - \sqrt{x}\big)\,dx = \int_{0}^{1} \big(2x^{1/3} - x^{1/2}\big)\,dx = \Big(\frac{3}{2}\,x^{4/3} - \frac{2}{3}\,x^{3/2}\Big)\Big|_{0}^{1} = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}.$$

Example 3: Evaluate $\displaystyle\int_{1}^{3} \frac{9}{x^4}\,dx$.

Solution: Converting to a negative exponent and remembering to add one to the exponent, we find that

$$\int_{1}^{3} \frac{9}{x^4}\,dx = \int_{1}^{3} 9\,x^{-4}\,dx = -3x^{-3}\Big|_{1}^{3} = -\frac{3}{x^3}\Big|_{1}^{3} = -\frac{1}{9} - (-3) = \frac{26}{9}.$$

Be careful with the negative numbers when computing $F(b) - F(a)$.

Example 4: We present three integrals. Carefully check each antiderivative.

$$\begin{aligned}\int_{0}^{1} \frac{8}{1+x}\,dx &= 8\ln|1+x|\Big|_{0}^{1} = 8\ln 2 - 8\ln 1 = 8\ln 2; \\ \int_{1}^{6} \frac{8}{1+4x}\,dx &= 2\ln|1+4x|\Big|_{1}^{6} = 2\ln 25 - 2\ln 5 = 2\ln 5; \\ \int_{0}^{1/2} \frac{8}{1+4x^2}\,dx &= 4\arctan(2x)\Big|_{0}^{1/2} = 4\arctan 1 - 4\arctan 0 = \pi.\end{aligned}$$

Extra Notes for Section 2.8

Example 1: Evaluate $\int \sin^3(\pi x)\cos(\pi x)\,dx$.

Solution: Since the sine function is raised to the third power, a reasonable guess is this same function to the fourth power. Using the chain rule, we have

$$\text{guess}: \ \sin^4(\pi x);$$
$$\text{check}: \ \tfrac{d}{dx}\sin^4(\pi x) = 4\sin^3(\pi x)\cos(\pi x)\cdot\pi = 4\pi\sin^3(\pi x)\cos(\pi x).$$

Since our guess is only off by a constant multiplier, we find that

$$\int \sin^3(\pi x)\cos(\pi x)\,dx = \frac{1}{4\pi}\ \sin^4(\pi x) + C.$$

Example 2: Evaluate $\int \sec^2(2x)\tan(2x)\,dx$.

Solution: We use the fact that the derivative of $\tan x$ is $\sec^2 x$ to inform our guess.

$$\text{guess}: \ \tan^2(2x);$$
$$\text{check}: \ \tfrac{d}{dx}\tan^2(2x) = 2\tan(2x)\sec^2(2x)\cdot 2 = 4\sec^2(2x)\tan(2x).$$

Since the derivative of our guess only differs from the integrand by a constant, we see that

$$\int \sec^2(2x)\tan(2x)\,dx = \frac{1}{4}\ \tan^2(2x) + C.$$

Note that $\int \sec^2(2x)\tan(2x)\,dx = \frac{1}{4}\ \sec^2(2x) + C$. Do you see how someone could arrive at this answer?

Example 3: Evaluate $\int \sec^3(4x)\tan(4x)\,dx$.

Solution: Unlike the previous problem, a guess of $\tan^2(4x)$ does not get us a constant multiple of the integrand. (We only get $\sec^2(4x)$ rather than $\sec^3(4x)$.) With a little thought, we remember that the derivative of $\sec x$ is $\sec x \tan x$. But even with this knowledge, we need to be careful since an extra secant shows up in the derivative. We thus arrive at the following:

$$\text{guess}: \ \sec^3(4x);$$
$$\text{check}: \ \tfrac{d}{dx}\sec^3(4x) = 3\sec^2(4x)\sec(4x)\tan(4x)\cdot 4 = 12\sec^3(4x)\tan(4x).$$

We thus find that

$$\int \sec^3(4x)\tan(4x)\,dx = \frac{1}{12}\ \sec^3(4x) + C.$$

Example 4: Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^4+9}}\,dx.$

Solution: As is often the case, we focus on the main portion of the integrand and hope that the chain rule takes care of the rest. Since x^4+9 is raised to the $-\frac{1}{2}$ power, we consider x^4+9 to the $-\frac{1}{2}+1=\frac{1}{2}$ power:

$$\begin{aligned}&\text{guess}: (x^4+9)^{1/2};\\&\text{check}: \tfrac{d}{dx}(x^4+9)^{1/2} = \frac{1}{2}(x^4+9)^{-1/2}\cdot 4x^3 = \frac{2x^3}{\sqrt{x^4+9}}.\end{aligned}$$

It follows that

$$\int \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{2}\sqrt{x^4+9}+C.$$

Example 5: Evaluate $\displaystyle\int \frac{x^3}{x^4+9}\,dx.$

Solution: We can do this one without a formal guess and check. Recognizing a function in the denominator raised to the first power with essentially its derivative in the numerator, we find that

$$\int \frac{x^3}{x^4+9}\,dx = \frac{1}{4}\ln(x^4+9)+C.$$

Example 6: Evaluate $\displaystyle\int \frac{x}{x^4+9}\,dx.$

Solution: Since the numerator does not involve an x^3 term, working with x^4+9 is not going to get us anywhere. Running through the basic antiderivative formulas, we eventually land in the $\arctan x$ camp. In order to get the x^4 term, we need to start with x^2.

$$\begin{aligned}&\text{guess}: \arctan(x^2/3);\\&\text{check}: \tfrac{d}{dx}\arctan(x^2/3) = \frac{2x/3}{1+(x^4/9)} = \frac{6x}{9+x^4}.\end{aligned}$$

It follows that

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{6}\arctan\Big(\frac{x^2}{3}\Big)+C.$$

Example 7: Evaluate $\displaystyle\int \frac{x^3}{x^2+9}\,dx.$

Solution: Since the degree of the polynomial in the numerator is greater than the degree of the polynomial in the denominator, we can use long division. However, we will just do some convenient algebra.

$$\int \frac{x^3}{x^2+9}\,dx = \int \frac{(x^3+9x)-9x}{x^2+9}\,dx = \int \Big(x-\frac{9x}{x^2+9}\Big)\,dx = \frac{1}{2}x^2-\frac{9}{2}\ln(x^2+9)+C.$$

Study each step carefully as this is a useful technique to know.

Extra Notes for Section 2.9

When presented with a function whose antiderivative is sought, remember to first think about basic formulas or slight modifications of these formulas. In such cases, you should just be able to write down the answer.

$$\int e^{-2x}\,dx = -\frac{1}{2}\,e^{-2x} + C;$$
$$\int 3e^{x/4}\,dx = 12\,e^{x/4} + C;$$
$$\int 6\cos(3x)\,dx = 2\sin(3x) + C;$$
$$\int \frac{3x}{2x^2+7}\,dx = \frac{3}{4}\,\ln(2x^2+7) + C;$$
$$\int \frac{2}{9+x^2}\,dx = \frac{2}{3}\,\arctan(x/3) + C.$$

Remember that you can check each of these solutions by taking the derivative of the proposed answer. If an antiderivative is not clear, you can try guess and check as in the previous section. However, sometimes a substitution makes the problem clearer.

Example 1: Evaluate $\displaystyle\int \frac{\sin x}{9+4\cos^2 x}\,dx$.

Solution: For this integral, since the denominator is raised to the first power, we might first think about the natural logarithm. However, the numerator is not a constant multiple of the derivative of the denominator so this idea is quickly ruled out. Next, we notice that there is a function squared in the denominator so we have a hint that perhaps an arctangent may be involved. When we realize that the function squared is $2\cos x$ and that its derivative $-2\sin x$ essentially appears in the numerator, we know we are on the right track. We proceed as follows:

$$\text{let } u = 2\cos x,$$
$$\text{then } du = -2\sin x\,dx;$$

and find that

$$\begin{aligned}\int \frac{\sin x}{9+4\cos^2 x}\,dx &= \int \frac{1}{9+u^2}\left(-\frac{1}{2}\right)du \\ &= -\frac{1}{2}\int \frac{1}{9+u^2}\,du \\ &= -\frac{1}{2}\cdot\frac{1}{3}\arctan(u/3) + C \\ &= -\frac{1}{6}\arctan\left(\frac{2}{3}\cos x\right) + C.\end{aligned}$$

For problems such as this, the substitution is not doing anything magical; it simply cleans things up so that we can recognize a basic formula.

Example 2: Evaluate $\displaystyle\int \frac{4x}{\sqrt{2x+5}}\,dx$.

Solution: In this case, we make the following substitution:

$$\begin{aligned} &\text{let } u = 2x+5; && 4x = 2(u-5);\\ &\text{then } du = 2\,dx; && dx = \frac{1}{2}\,du; \end{aligned}$$

and obtain

$$\begin{aligned}
\int \frac{4x}{\sqrt{2x+5}}\,dx &= \int \frac{2(u-5)}{\sqrt{u}}\left(\frac{1}{2}\right)du\\
&= \int \frac{u-5}{\sqrt{u}}\,du\\
&= \int \left(u^{1/2} - 5u^{-1/2}\right)du\\
&= \frac{2}{3}\,u^{3/2} - 10u^{1/2} + C\\
&= \frac{2}{3}\,(2x+5)^{3/2} - 10(2x+5)^{1/2} + C\\
&= \frac{2}{3}\,(2x+5)^{1/2}\big((2x+5) - 15\big) + C\\
&= \frac{4}{3}\,(2x+5)^{1/2}(x-5) + C.
\end{aligned}$$

The last two steps are not necessary. They do illustrate some algebra techniques that can sometimes be helpful and indicate that a correct answer can assume several different forms.

Example 3: Evaluate $\displaystyle\int_0^4 \frac{1}{9-\sqrt{x}}\,dx$.

Solution: Since this is a definite integral, it is best to change the limits of integration to the new variable and never return to the original variable. Although it is not obvious, we try the following:

$$\begin{aligned} &\text{let } u = 9-\sqrt{x}; && u = 9 \text{ when } x = 0;\\ &\text{then } x = (9-u)^2 \text{ and } dx = -2(9-u)\,du; && u = 7 \text{ when } x = 4; \end{aligned}$$

$$\int_0^4 \frac{1}{9-\sqrt{x}}\,dx = \int_9^7 \frac{-2(9-u)}{u}\,du = 2\int_7^9 \left(\frac{9}{u} - 1\right)du = 2\Big(9\ln u - u\Big)\Big|_7^9 = 18\ln(9/7) - 4.$$

As a reminder of our previous work in Section 2.5, note that

$$\frac{1}{9} \le \frac{1}{9-\sqrt{x}} \le \frac{1}{7}$$

for all $x \in [0,4]$ and this implies that the value of the integral is between $4/9$ and $4/7$. Using a calculator, we find that

$$0.444444 \approx \frac{4}{9} < 0.523660 \approx 18\ln(9/7) - 4 < \frac{4}{7} \approx 0.571429.$$

Our rough estimate thus shows that our final answer is reasonable.

Extra Notes for Section 2.10

Example 1: Evaluate $\int xe^{kx}\,dx$, where k is a constant.

Solution: We proceed with integration by parts.

$$\int xe^{kx}\,dx, \qquad \begin{array}{lcll} \text{let} & u = x & \text{and} & dv = e^{kx}\,dx; \\ \text{then} & du = dx & \text{and} & v = \frac{1}{k}\,e^{kx}; \end{array}$$

$$\int xe^{kx}\,dx = \frac{1}{k}\,xe^{kx} - \int \frac{1}{k}\,e^{kx}\,dx = \frac{1}{k}\,xe^{kx} - \frac{1}{k^2}\,e^{kx} + C = \frac{1}{k^2}(kx-1)e^{kx} + C.$$

Example 2: Evaluate $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$.

Solution: This problem appeared in the last section (problem 1i with $a = 2$) so it can be done using a substitution. However, it can also be solved using integration by parts. Here are a couple of options.

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx, \qquad \begin{array}{lcll} \text{let} & u = \frac{1}{\sqrt{4-x^2}} & \text{and} & dv = x^3\,dx; \\ \text{then} & du = \frac{x}{(4-x^2)^{3/2}}\,dx & \text{and} & v = \frac{1}{4}\,x^4; \end{array}$$

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx, \qquad \begin{array}{lcll} \text{let} & u = x^3 & \text{and} & dv = \frac{1}{\sqrt{4-x^2}}\,dx; \\ \text{then} & du = 3x^2\,dx & \text{and} & v = \arcsin(x/2); \end{array}$$

For both of these cases, the new integral $\int v\,du$ looks at least as complicated, if not more so, than the original problem. This indicates that we need to make a different choice. Eventually, we are led to the following, where we put one of the x's with the square root in dv and leave an x^2 with u.

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx, \qquad \begin{array}{lcll} \text{let} & u = x^2 & \text{and} & dv = \frac{x}{\sqrt{4-x^2}}\,dx; \\ \text{then} & du = 2x\,dx & \text{and} & v = -\sqrt{4-x^2}; \end{array}$$

We then have

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = -x^2\sqrt{4-x^2} - \int \sqrt{4-x^2}\,(-2x)\,dx = -x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{3/2} + C$$

This is a suitable answer but we can also simplify it using some algebra.

$$-x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{3/2} + C = -\frac{1}{3}\sqrt{4-x^2}\left(3x^2 + 8 - 2x^2\right) + C = -\frac{1}{3}\,(x^2+8)\sqrt{4-x^2} + C$$

Then (to give an example of a definite integral), we find that

$$\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{4-x^2}}\,dx = -\frac{1}{3}\,(x^2+8)\sqrt{4-x^2}\Big|_0^{\sqrt{3}} = -\frac{1}{3}\,(11-16) = \frac{5}{3}.$$

Example 3: Evaluate $\int x^2 e^{kx}\,dx$, where k is a constant.

Solution: We proceed with integration by parts.

$$\int x^2 e^{kx}\,dx, \qquad \begin{array}{lccl} \text{let} & u = x^2 & \text{and} & dv = e^{kx}\,dx; \\ \text{then} & du = 2x\,dx & \text{and} & v = \dfrac{1}{k}\,e^{kx}; \end{array}$$

$$\int x^2 e^{kx}\,dx = \frac{1}{k}\,x^2 e^{kx} - \frac{2}{k}\int x e^{kx}\,dx$$

The new integral is easier than the original integral so we are moving in the right direction. At this stage, we would have to use integration by parts one more time. However, in this case, we can refer to Example 1 and obtain

$$\int x^2 e^{kx}\,dx = \frac{1}{k}\,x^2 e^{kx} - \frac{2}{k}\cdot\frac{1}{k^2}(kx-1)e^{kx} + C = \frac{1}{k^3}\big(k^2x^2 - 2kx + 2\big)e^{kx} + C$$

The advantage of using a generic value like k is that you end up with a general formula that you can use. For example, with $k = -1/2$, we have

$$\int x^2 e^{-x/2}\,dx = -8\Big(\frac{1}{4}\,x^2 + x + 2\Big)e^{-x/2} + C = \big(-2x^2 - 8x - 16\big)e^{-x/2} + C$$

Example 4: Evaluate $\int e^x \sin x\,dx$.

Solution: This is another classic example that involves integration by parts.

$$\int e^x \sin x\,dx, \qquad \begin{array}{lccl} \text{let} & u = e^x & \text{and} & dv = \sin x\,dx; \\ \text{then} & du = e^x\,dx & \text{and} & v = -\cos x; \end{array}$$

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

It appears that we have made no progress but we press on with the new integral.

$$\int e^x \cos x\,dx, \qquad \begin{array}{lccl} \text{let} & u = e^x & \text{and} & dv = \cos x\,dx; \\ \text{then} & du = e^x\,dx & \text{and} & v = \sin x; \end{array}$$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

It seems we are moving in circles. However, putting the pieces together, we find that

$$\begin{aligned} \int e^x \sin x\,dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx; \\ 2\int e^x \sin x\,dx &= -e^x \cos x + e^x \sin x; \\ \int e^x \sin x\,dx &= \frac{1}{2}\big(\sin x - \cos x\big)e^x + C. \end{aligned}$$

You should check this answer by taking the derivative of the right-hand side.

Extra Notes for Section 2.11

Example 1: Evaluate $\displaystyle\int_2^\infty \frac{6}{x^4}\,dx$.

Solution: Since this is an improper integral, we need to take a limit as the upper limit of integration goes to infinity:

$$\begin{aligned}\int_2^\infty \frac{6}{x^4}\,dx &= \lim_{b\to\infty}\int_2^b \frac{6}{x^4}\,dx \\ &= \lim_{b\to\infty} \frac{-2}{x^3}\bigg|_2^b \\ &= \lim_{b\to\infty}\left(\frac{-2}{b^3}+\frac{1}{4}\right) \\ &= \frac{1}{4}.\end{aligned}$$

Example 2: Evaluate $\displaystyle\int_0^\infty \frac{e^x+e^{2x}}{e^{3x}}\,dx$.

Solution: We proceed as in the previous example, using division as the first step in the integration process.

$$\begin{aligned}\int_0^\infty \frac{e^x+e^{2x}}{e^{3x}}\,dx &= \lim_{b\to\infty}\int_0^b \left(e^{-2x}+e^{-x}\right)dx \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}\,e^{-2x}-e^{-x}\right)\bigg|_0^b \\ &= \lim_{b\to\infty}\left(\left(-\frac{1}{2}\,e^{-2b}-e^{-b}\right)-\left(-\frac{1}{2}-1\right)\right) \\ &= \lim_{b\to\infty}\left(\frac{3}{2}-\frac{1}{2e^{2b}}-\frac{1}{e^b}\right) \\ &= \frac{3}{2}.\end{aligned}$$

Example 3: Evaluate $\displaystyle\int_1^\infty \frac{1}{\sqrt[6]{x^5}}\,dx$.

Solution: We evaluate this improper integral in the usual way.

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt[6]{x^5}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt[6]{x^5}}\,dx \\ &= \lim_{b\to\infty} 6\sqrt[6]{x}\,\bigg|_1^b \\ &= \lim_{b\to\infty}\left(6\sqrt[6]{b}-6\right) \\ &= \infty.\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

Example 4: Evaluate $\int_0^\infty xe^{-x/3}\,dx$.

Solution: We proceed in the usual way, noting that integration by parts is needed to find an antiderivative ($u = x$, $du = dx$, $dv = e^{-x/3}\,dx$, $v = -3e^{-x/3}$).

$$\begin{aligned}\int_0^\infty xe^{-x/3}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x/3}\,dx \\ &= \lim_{b\to\infty}\left(-3xe^{-x/3}\Big|_0^b + \int_0^b 3e^{-x/3}\,dx\right) \\ &= \lim_{b\to\infty}\left(-3be^{-b/3} - 9e^{-x/3}\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(-3be^{-b/3} - 9e^{-b/3} + 9\right) \\ &= \lim_{b\to\infty}\left(9 - \frac{3b+9}{e^{b/3}}\right) = 9.\end{aligned}$$

Example 5: Use Theorem 2.6 to prove that the integral $\int_0^\infty e^{-x}\cos^2(x^2)\,dx$ converges.

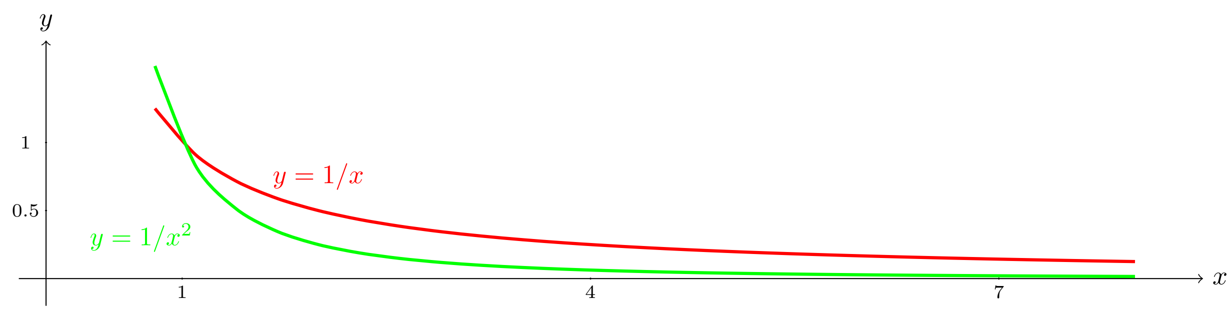
Solution: We begin by noting that $0 \le e^{-x}\cos^2(x^2) \le e^{-x}$ for all $x \ge 0$ since the value of the cosine function is always between -1 and 1. Since

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty}\left(-e^{-b}+1\right) = 1,$$

the improper integral $\int_0^\infty e^{-x}\,dx$ converges. It follows from Theorem 2.6 that $\int_0^\infty e^{-x}\cos^2(x^2)\,dx$ converges.

Example 6: Compare the areas under the curves $y = 1/x$ and $y = 1/x^2$ on the interval $[1,\infty)$.

Solution: The two curves are sketched below. Since areas can be determined by integrals, the areas under the curves can be represented by $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$.



Evaluating these improper integrals yields

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln x\Big|_1^b = \lim_{b\to\infty}\ln b = \infty;$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$

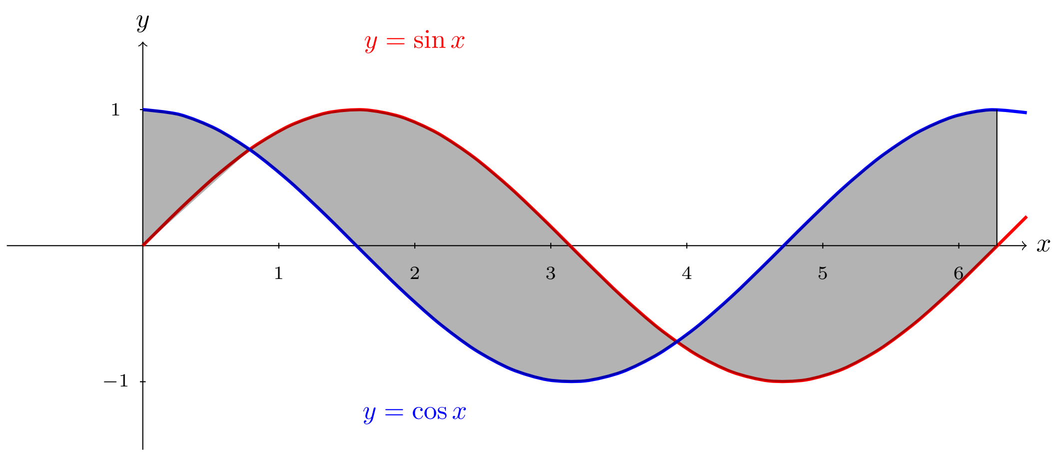
It follows that the area under the graph of $y = 1/x$ on the interval $[1,\infty)$ is infinite while the area under the graph of $y = 1/x^2$ on the interval $[1,\infty)$ has a value of 1. Since the graphs of the two functions appear similar, the dramatic difference between the areas is quite surprising.

Extra Notes for Section 2.12

For area problems, it is best to begin with a careful sketch of the region whose area is sought. Most of the functions that we consider in calculus have graphs that should be familiar to you. If necessary, you can use a calculator to obtain the graph. The next task is to find the points where the curves intersect. Again, for most of the problems we consider, these can be found by solving a simple equation. The last step before integrating is to determine whether it is best to describe the region vertically or horizontally. If you take the vertical approach, then you want to solve for y in terms of x and take the upper curve minus the lower curve. If you take the horizontal approach, then you want to solve for x in terms of y and take the right curve minus the left curve.

Example 1: Find the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$.

Solution: The region is sketched below. Since the curves cross each other several times, we need to be careful setting up the integrals. Note that the curves meet when $x = \pi/4$ and $x = 5\pi/4$.

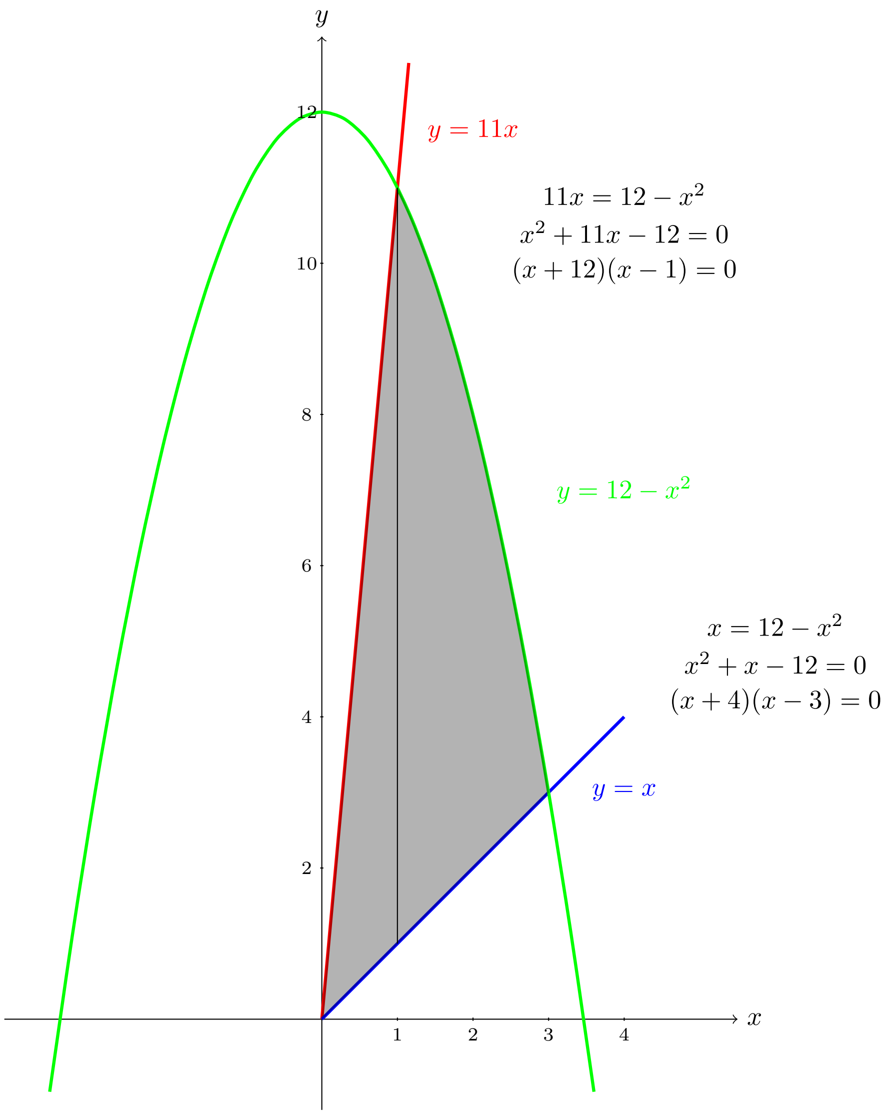


The area A of this region is thus

$$\begin{aligned}
A &= \int_0^{2\pi} \left|\cos x - \sin x\right| dx \\
&= \int_0^{\pi/4} \big(\cos x - \sin x\big)\, dx + \int_{\pi/4}^{5\pi/4} \big(\sin x - \cos x\big)\, dx + \int_{5\pi/4}^{2\pi} \big(\cos x - \sin x\big)\, dx \\
&= \big(\sin x + \cos x\big)\Big|_0^{\pi/4} + \big(-\cos x - \sin x\big)\Big|_{\pi/4}^{5\pi/4} + \big(\sin x + \cos x\big)\Big|_{5\pi/4}^{2\pi} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right) + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) + \left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \\
&= 4\sqrt{2}.
\end{aligned}$$

Example 2: Find the area of the region in the first quadrant that is bounded by the curves $y = x$, $y = 11x$, and $y = 12 - x^2$.

Solution: The region of interest is sketched below.



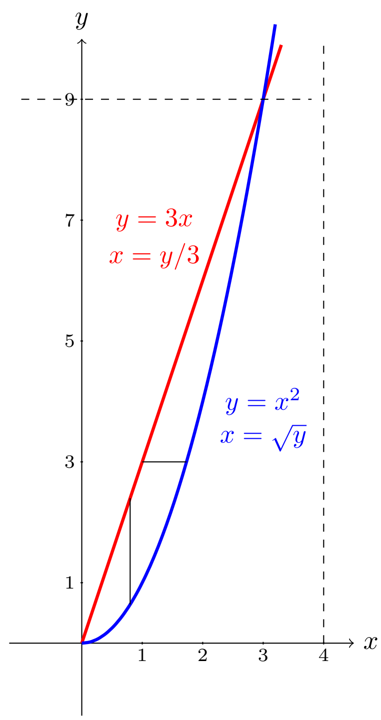
The points of intersection are determined by setting the appropriate curves equal to each other. From the figure, we see that the upper boundary of the region changes at $x = 1$. It follows that

$$A = \int_0^1 (11x - x)\,dx + \int_1^3 \big((12 - x^2) - x\big)\,dx = \int_0^1 10x\,dx + \int_1^3 (12 - x - x^2)\,dx$$
$$= 5x^2\Big|_0^1 + \left(12x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_1^3 = 5 + 24 - 4 - \frac{26}{3} = \frac{49}{3}.$$

Hence, the area of the shaded region is 49/3 square units.

Example 1: Let R be the region bounded by the curves $y = 3x$ and $y = x^2$. Set up an integral that represents the volume of the solid that is generated when R is revolved around (a) the x-axis, (b) the y-axis, (c) the line $x = 4$, or (d) the line $y = 9$. Then set up an integral for the volume of the solid whose base is R and (e) each cross-section perpendicular to the y-axis is a square or (f) each cross-section perpendicular to the x-axis is a semicircle.

Solution: The region R is sketched below.



For the first four volumes, we use the washer method. Identifying the larger and smaller radius in each case, we find that

$$V_a = \int_0^3 \Big(\pi(3x)^2 - \pi(x^2)^2\Big)\,dx;$$

$$V_b = \int_0^9 \left(\pi\big(\sqrt{y}\big)^2 - \pi\Big(\frac{y}{3}\Big)^2\right)dy;$$

$$V_c = \int_0^9 \left(\pi\Big(4 - \frac{y}{3}\Big)^2 - \pi\big(4 - \sqrt{y}\big)^2\right)dy;$$

$$V_d = \int_0^3 \Big(\pi(9 - x^2)^2 - \pi(9 - 3x)^2\Big)\,dx.$$

The last two problems are a bit different since the solids are not generated by revolution. For part (e), we look at the horizontal distance across the figure (since the cross-sections are taken perpendicular to the

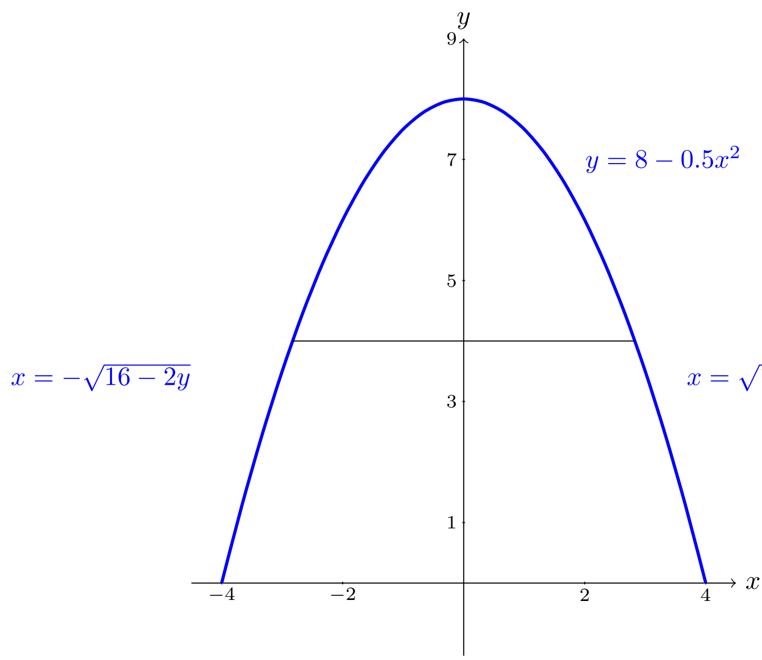
y-axis). This distance is given by $\sqrt{y} - y/3$ and represents the length of one side of the square. For part (f), we look at the vertical distance across the figure (since the cross-sections are taken perpendicular to the x-axis). This distance is given by $3x - x^2$ and represents the length of the diameter of the semicircle. Using the familiar formulas for the area of a square and the area of a semicircle, it follows that

$$V_e = \int_0^9 \left(\sqrt{y} - \frac{y}{3}\right)^2 dy \qquad \text{and} \qquad V_f = \int_0^3 \frac{\pi}{2}\left(\frac{3x - x^2}{2}\right)^2 dx.$$

In each of these six cases, the value of the integral can easily be determined by multiplying out the integrand. In case you are interested, the values are $162\pi/5$, $27\pi/2$, $45\pi/2$, $243\pi/5$, $27/10$, and $81\pi/80$, respectively. You might find it helpful to ponder why V_a is larger than V_b.

Example 2: Let R be the region that lies beneath the curve $y = 8 - 0.5x^2$ and above the x-axis. Suppose that R is the base of a cave and that each cross-section perpendicular to the y-axis is a semicircle. Find the volume of the cave.

Solution: The region R is sketched below. The horizontal line represents the diameter of the semicircular cross-section.



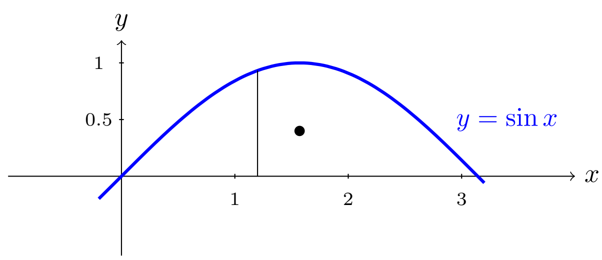
Since the cross-sections are taken in the horizontal direction, we need to solve the equation for x in terms of y; note that the square root gives us two solutions, one representing the curve on the right and the other the curve on the left. The horizontal distance is thus $2\sqrt{16 - 2y}$ and it follows that the radius of the semicircle is $\sqrt{16 - 2y}$. Using the formula for the area of a semicircle, the volume V of the cave is

$$V = \int_0^8 \frac{1}{2}\,\pi\left(\sqrt{16 - 2y}\right)^2 dy = \pi \int_0^8 (8 - y)\, dy = \pi\left(8y - \frac{1}{2}\,y^2\right)\Big|_0^8 = 32\pi.$$

Extra Notes for Section 2.14

Example 1: Let R be the region that lies beneath the curve $y = \sin x$ and above the x-axis on the interval $[0, \pi]$. Find the volume of the solid that is generated when R is revolved around (a) the x-axis and (b) the y-axis.

Solution: The region R is sketched below. Since the cross-sections of this curve are most easily represented vertically, we consider vertical cross-sections and then adapt our methods as necessary.



To determine the volume that is generated when R is revolved around the x-axis, our vertical cross-sections generate disks. Using the disk method, we find that the volume V_x is given by

$$V_x = \int_0^\pi \pi \sin^2 x \, dx = \frac{\pi}{2} \int_0^\pi \big(1 - \cos(2x)\big) \, dx = \frac{\pi}{2} \Big(x - \frac{1}{2} \sin(2x) \Big) \Big|_0^\pi = \frac{\pi^2}{2}.$$

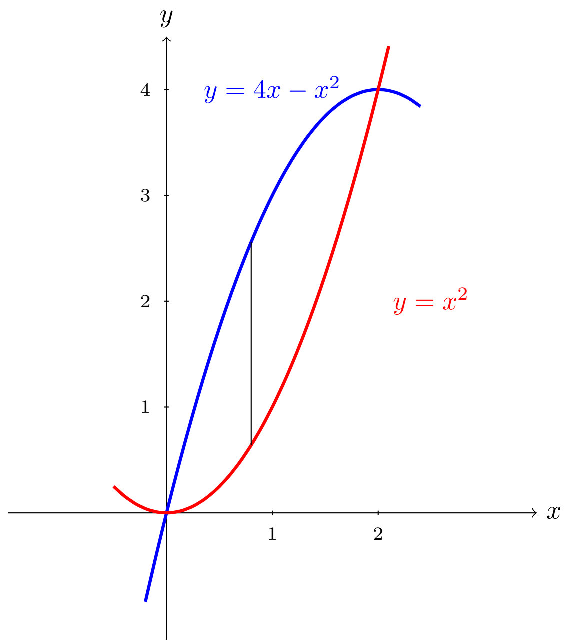
Note the use of the half-angle formula for sine; it is a good idea to be aware of this trigonometric identity (see Section 1.16). To determine the volume that is generated when R is revolved around the y-axis, we still want to use our vertical cross-sections but now they generate shells. Using the shell method, we find that the volume V_y is given by

$$V_y = \int_0^\pi 2\pi x \sin x \, dx = 2\pi \int_0^\pi x \sin x \, dx = 2\pi \big(-x \cos x + \sin x \big) \Big|_0^\pi = 2\pi^2.$$

The bullet in the figure represents the center of mass of R. Note that the center of mass moves further when R is revolved around the y-axis than it does when R is revolved around the x-axis. As a result, the value V_y is greater than V_x (see Section 2.18).

Example 2: Let R be the region bounded by the curves $y = x^2$ and $y = 4x - x^2$. Set up an integral that represents the volume of the solid that is generated when R is revolved around (a) the x-axis, (b) the y-axis, (c) the line $x = 4$, (d) the line $x = -3$, or (e) the line $y = 5$.

Solution: The region R is sketched below. Since the equation $y = 4x - x^2$ is a bit messy to solve for x in terms of y, it is easiest to describe the region with vertical cross-sections. This means that we want to use cross-sections (which will be washers) when R is revolved around any line parallel to the x-axis and shells when R is revolved around any line parallel to the y-axis. In the case of washers, we need to identify the outer radius and the inner radius. For shells, we just need to adjust the radius of the circular path through which the skinny vertical rectangle moves.



We thus obtain the following integrals for each of the requested volumes.

$$V_b = \int_0^2 2\pi x\big(4x - 2x^2\big)\,dx;$$
$$V_c = \int_0^2 2\pi(4-x)\big(4x - 2x^2\big)\,dx;$$
$$V_d = \int_0^2 2\pi(x+3)\big(4x - 2x^2\big)\,dx;$$
$$V_a = \int_0^2 \Big(\pi(4x - x^2)^2 - \pi(x^2)^2\Big)\,dx;$$
$$V_e = \int_0^2 \Big(\pi(5 - x^2)^2 - \pi\big(5 - (4x - x^2)\big)^2\Big)\,dx.$$

Example 3: Let R be the region bounded by the curves $y = x^2$ and $y = 4x - x^2$. Set up an integral to find the volume of the solid whose base is R and each cross-section perpendicular to the y-axis is a square.

Solution: To find this volume, we must use horizontal cross-sections (see the graph above) so it is necessary to solve the equations for x (completing the square for the second equation):

$$y = x^2 \;\Leftrightarrow\; x = \pm\sqrt{y} \quad \text{and} \quad y = 4x - x^2 \;\Leftrightarrow\; x^2 - 4x + 4 = 4 - y \;\Leftrightarrow\; (x-2)^2 = 4 - y \;\Leftrightarrow\; x = 2 \pm \sqrt{4-y}.$$

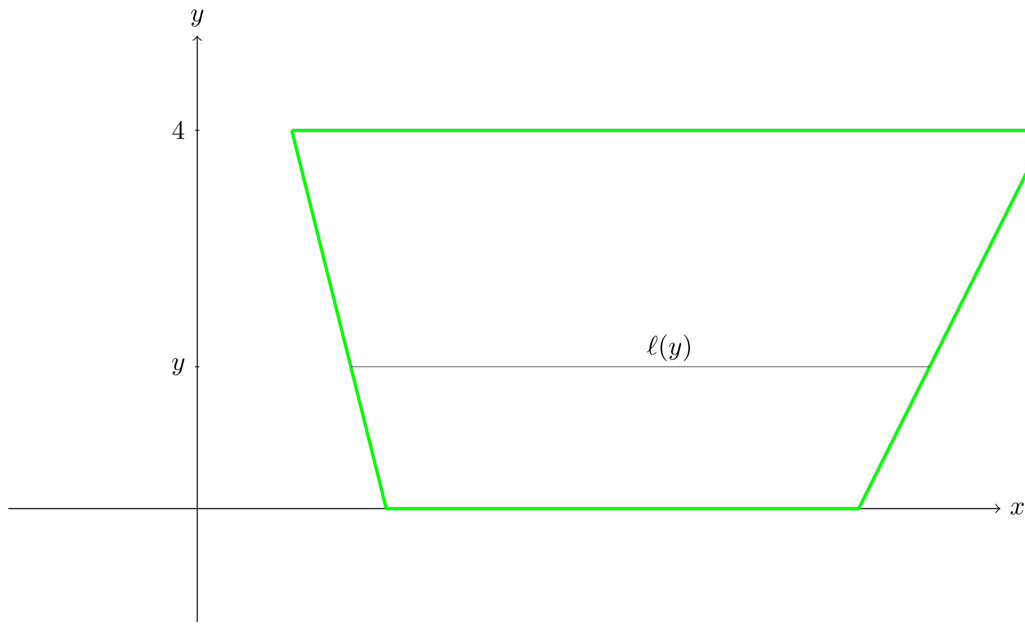
To obtain the correct curves, we choose the plus sign in the first case and the minus sign in the second case. The side of the square is the horizontal distance across the figure. Subtracting the left curve from the right curve to find this side then squaring the result to obtain the area of the square cross-section yields

$$V = \int_0^4 \Big(\sqrt{y} - \big(2 - \sqrt{4-y}\,\big)\Big)^2\,dy.$$

Extra Notes for Section 2.15

Example 1: Find the force exerted by a liquid with weight density w on one side of the vertically submerged trapezoidal plate shown below. The height of the trapezoid is 4 feet, the upper base is 8 feet, and the lower base is 5 feet. Assume that the top of the plate is 6 feet below the surface of the liquid.

Solution: We place a coordinate system so that the lower base of the trapezoid is on the x-axis. It then follows that the upper base of the trapezoid is on the line $y = 4$ and the level of the liquid is the line $y = 10$.



To find the length across the figure, we do not need to find the equations of the two lines forming the sides of the trapezoid. We know that the distance $\ell(y)$ is a linear function (since each of the sides is a straight line) that satisfies $\ell(0) = 5$ and $\ell(4) = 8$. If we view the coordinates of this line as (y, ℓ), then the line passes through the points $(0, 5)$ and $(4, 8)$. Using the point-slope form of a line, it follows that

$$\ell - 5 = \frac{8-5}{4-0}\,(y - 0) \quad \text{or} \quad \ell = 5 + \frac{3}{4}\,y.$$

(You may be able to find the function $\ell(y) = 5 + \frac{3}{4}\,y$ by inspection and that is fine.) The force F (in pounds) exerted by the liquid on the plate is thus

$$\begin{aligned} F &= \int_0^4 w(10-y)\Big(5 + \frac{3}{4}\,y\Big)\,dy = \frac{w}{4}\int_0^4 (10-y)(20+3y)\,dy \\ &= \frac{w}{4}\int_0^4 (200 + 10y - 3y^2)\,dy = \frac{w}{4}\Big(200y + 5y^2 - y^3\Big)\Big|_0^4 \\ &= \frac{w}{4}\big(800 + 80 - 64\big) = w\big(200 + 20 - 16\big) = 204w. \end{aligned}$$

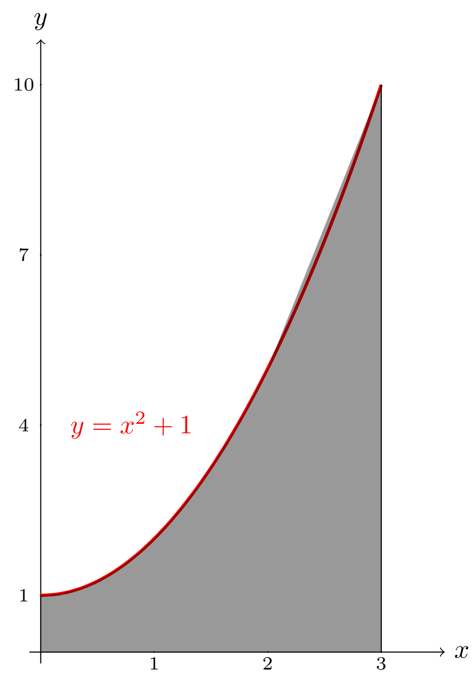
Extra Notes for Section 2.16

Since this section is rarely covered (due to lack of time in the semester as well as the need for some physics background which many students may not have), there are no extra notes for this section at this time.

Extra Notes for Section 2.17

Example 1: Find the center of mass for the region R that lies below the curve $y = x^2 + 1$ and above the x-axis on the interval $[0, 3]$. Also, find the center of mass of the solid S that is generated when R is revolved around the x-axis. Assume that the density is constant in both cases.

Solution: The region R is shown in the figure. Letting ρ denote the density (mass per area) for R, we find that



$$\int dm = \int_0^3 \rho(x^2+1)\,dx = \rho\Big(\frac{1}{3}\,x^3 + x\Big)\Big|_0^3 = 12\rho;$$

$$\int x\,dm = \int_0^3 x\,\rho(x^2+1)\,dx = \rho\Big(\frac{1}{4}\,x^4 + \frac{1}{2}\,x^2\Big)\Big|_0^3 = \frac{99}{4}\,\rho;$$

$$\int y\,dm = \int_0^1 y\rho\,3\,dy + \int_1^{10} y\,\rho\big(3 - \sqrt{y-1}\,\big)\,dy = \rho\left(\int_0^{10} 3y\,dy - \int_1^{10} y\,\sqrt{y-1}\,dy\right)$$

$$= \rho\left(150 - \int_0^9 (u+1)\,\sqrt{u}\,du\right) = \rho\Big(150 - \frac{2}{5}\cdot 9^{5/2} - \frac{2}{3}\cdot 9^{3/2}\Big) = \rho\Big(150 - \frac{486}{5} - 18\Big) = 34.8\,\rho.$$

It follows that $\overline{x} = 33/16$ and $\overline{y} = 2.9$ for the region R. Using ρ_1 for the density (mass per volume) of the solid S, we find that

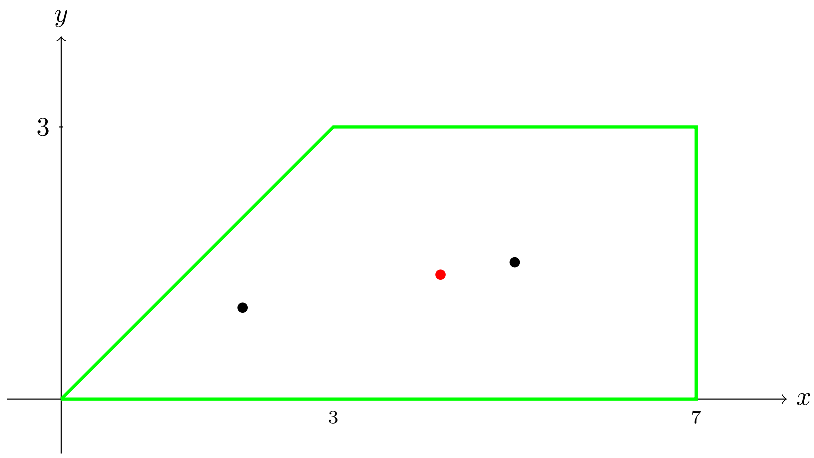
$$\int dm = \int_0^3 \rho_1\pi(x^2+1)^2\,dx = \rho_1\pi\Big(\frac{1}{5}\,x^5 + \frac{2}{3}\,x^3 + x\Big)\Big|_0^3 = \frac{348}{5}\,\pi\rho_1;$$

$$\int x\,dm = \int_0^3 x\,\rho_1\pi(x^2+1)^2\,dx = \rho_1\pi\Big(\frac{1}{6}\,x^6 + \frac{2}{4}\,x^4 + \frac{1}{2}\,x^2\Big)\Big|_0^3 = \frac{333}{2}\,\pi\rho_1.$$

It then follows that the x-coordinate of the center of mass of S is $555/232 \approx 2.39$. By symmetry, we find that $\overline{y} = 0 = \overline{z}$.

Example 2: Find the centroid of the trapezoid with vertices $(0, 0)$, $(3, 3)$, $(7, 3)$, and $(7, 0)$.

Solution: Since we are seeking the centroid, we may assume that the constant density of the region has the value 1. A sketch of the trapezoid is given below.



In this case, we find that

$$\int dm = \text{area of region } = 4.5 + 12 = 16.5;$$
$$\int x\,dm = \int_0^3 x(x\,dx) + \int_3^7 x(3\,dx) = 9 + 60 = 69;$$
$$\int y\,dm = \int_0^3 y\big((7-y)\,dy\big) = \frac{7}{2}\cdot 9 - \frac{1}{3}\cdot 27 = 22.5.$$

It follows that

$$\overline{x} = \frac{\displaystyle\int x\,dm}{\displaystyle\int dm} = \frac{69}{16.5} = \frac{23}{5.5} = \frac{46}{11} \quad \text{and} \quad \overline{y} = \frac{\displaystyle\int y\,dm}{\displaystyle\int dm} = \frac{22.5}{16.5} = \frac{7.5}{5.5} = \frac{15}{11}.$$

The centroid is represented by the red dot in the figure. For the record, the black dots in the figure represent the center of mass of the right triangle (at $(2, 1)$) and the rectangle (at $(5, 1.5)$). If we place a mass of 4.5 (the area of the triangle) at $(2, 1)$ and a mass of 12 (the area of the rectangle) at $(5, 1.5)$, the center of mass of this pair of objects will be at the red dot:

$$\overline{x} = \frac{\displaystyle\int x\,dm}{\displaystyle\int dm} = \frac{2\cdot 4.5 + 5\cdot 12}{4.5+12} = \frac{69}{16.5} = \frac{46}{11} \quad \text{and} \quad \overline{y} = \frac{\displaystyle\int y\,dm}{\displaystyle\int dm} = \frac{1\cdot 4.5 + 1.5\cdot 12}{4.5+12} = \frac{22.5}{16.5} = \frac{15}{11}.$$

The above calculation indicates how the generic formulas for the coordinates of the center of mass may be finite sums.

Extra Notes for Section 2.19

The main thrust of this section is to use algebra and the basic formulas

$$\int \frac{du}{u} = \ln|u| + C, \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\,\arctan(u/a) + C, \qquad \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin(u/a) + C,$$

to evaluate a variety of integrals. As a start, we note that

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\,\ln(x^2+16) + C; \qquad \int \frac{x}{\sqrt{16-x^2}}\,dx = -\sqrt{16-x^2} + C;$$
$$\int \frac{1}{x^2+16}\,dx = \frac{1}{4}\,\arctan(x/4) + C; \qquad \int \frac{1}{\sqrt{16-x^2}}\,dx = \arcsin(x/4) + C;$$
$$\int \frac{x}{4x^2+9}\,dx = \frac{1}{8}\,\ln(4x^2+9) + C; \qquad \int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{4}\,\sqrt{9-4x^2} + C;$$
$$\int \frac{1}{4x^2+9}\,dx = \frac{1}{6}\,\arctan(2x/3) + C; \qquad \int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\,\arcsin(2x/3) + C.$$

(We can omit the absolute values for the natural logarithm in these cases since the function in question is always positive.) You should be able to check the antiderivatives involving the natural logarithm or the square root mentally; an informal version of guess and check. To illustrate the second arctangent result, note that

$$\int \frac{1}{4x^2+9}\,dx = \frac{1}{2}\int \frac{2\,dx}{3^2+(2x)^2} = \frac{1}{2}\cdot\frac{1}{3}\,\arctan(2x/3) + C = \frac{1}{6}\,\arctan(2x/3) + C;$$

the first step rearranges the integrand so that it looks exactly like the formula with $a = 3$ and $u = 2x$. It is then important to make sure that the du term is correct; it is $2\,dx$ in this case. Using these ideas and some elementary algebra, we find that

$$\int \frac{3x+7}{x^2+16}\,dx = \int \Big(\frac{3x}{x^2+16} + \frac{7}{x^2+16}\Big)\,dx = \frac{3}{2}\,\ln(x^2+16) + \frac{7}{4}\,\arctan(x/4) + C;$$
$$\int \frac{3x+7}{\sqrt{16-x^2}}\,dx = \int \Big(\frac{3x}{\sqrt{16-x^2}} + \frac{7}{\sqrt{16-x^2}}\Big)\,dx = -3\sqrt{16-x^2} + 7\arcsin(x/4) + C.$$

Here are a few more examples that involve some elementary algebra; ponder the details carefully.

$$\int \frac{x}{2x-5}\,dx = \frac{1}{2}\int \frac{2x-5+5}{2x-5}\,dx = \frac{1}{2}\int \Big(1 + \frac{5}{2x-5}\Big)\,dx = \frac{1}{2}\Big(x + \frac{5}{2}\ln|2x-5|\Big) + C;$$

$$\int \frac{x^2}{x+1}\,dx = \int \frac{x^2-1+1}{x+1}\,dx = \int \Big(x - 1 + \frac{1}{x+1}\Big)\,dx = \frac{1}{2}(x-1)^2 + \ln|x+1| + C;$$

$$\int \frac{x^3+2x^2-3x+1}{x^2+2}\,dx = \int \Big(x + 2 - \frac{5x+3}{x^2+2}\Big)\,dx = \int \Big(x + 2 - \frac{5x}{x^2+2} - \frac{3}{x^2+2}\Big)\,dx$$
$$= \frac{1}{2}\,x^2 + 2x - \frac{5}{2}\,\ln(x^2+2) - \frac{3}{\sqrt{2}}\,\arctan\big(x/\sqrt{2}\,\big) + C.$$

For the next two examples, we use the technique of completing the square.

Example 1: Evaluate $\displaystyle\int \frac{2x+5}{x^2-8x+30}\,dx$.

Solution:

$$\begin{aligned}
\int \frac{2x+5}{x^2-8x+30}\,dx &= \int \frac{2x+5}{(x-4)^2+14}\,dx && \text{complete the square}\\
&= \int \frac{2(u+4)+5}{u^2+14}\,du && \text{let } u = x-4\\
&= \int \Big(\frac{2u}{u^2+14} + \frac{13}{u^2+14}\Big)\,du && \text{split up}\\
&= \ln|u^2+14| + \frac{13}{\sqrt{14}}\,\arctan\big(u/\sqrt{14}\,\big) + C && \text{basic formulas}\\
&= \ln|x^2-8x+30| + \frac{13}{\sqrt{14}}\,\arctan\Big(\frac{x-4}{\sqrt{14}}\Big) + C && \text{return to } x
\end{aligned}$$

Example 2: Evaluate $\displaystyle\int \frac{3x-1}{\sqrt{7-4x-x^2}}\,dx$.

Solution: We first note that

$$7-4x-x^2 = -(x^2+4x)+7 = -(x^2+4x+4)+4+7 = 11-(x+2)^2.$$

It then follows that

$$\begin{aligned}
\int \frac{3x-1}{\sqrt{7-4x-x^2}}\,dx &= \int \frac{3x-1}{\sqrt{11-(x+2)^2}}\,dx && \text{complete the square}\\
&= \int \frac{3(u-2)-1}{\sqrt{11-u^2}}\,du && \text{let } u = x+2\\
&= \int \Big(\frac{3u}{\sqrt{11-u^2}} - \frac{7}{\sqrt{11-u^2}}\Big)\,du && \text{split up}\\
&= -3\sqrt{11-u^2} - 7\arcsin\big(u/\sqrt{11}\,\big) + C && \text{basic formulas}\\
&= -3\sqrt{7-4x-x^2} - 7\arcsin\Big(\frac{x+2}{\sqrt{11}}\Big) + C && \text{return to } x
\end{aligned}$$

Example 3: Evaluate $\displaystyle\int \frac{x^3+5}{x^2+9}\,dx$.

Solution: One approach to this problem, which should be used if you are not confident in your algebra skills, is to use long division. Here we illustrate some creative algebra.

$$\begin{aligned}
\int \frac{x^3+5}{x^2+9}\,dx &= \int \frac{x^3+9x-9x+5}{x^2+9}\,dx && \text{add 0 in a convenient way}\\
&= \int \frac{x(x^2+9)-9x+5}{x^2+9}\,dx && \text{factor}\\
&= \int \Big(x - \frac{9x}{x^2+9} + \frac{5}{x^2+9}\Big)\,dx && \text{split up}\\
&= \frac{1}{2}\,x^2 - \frac{9}{2}\,\ln(x^2+9) + \frac{5}{3}\,\arctan\big(x/3\big) + C && \text{basic formulas}
\end{aligned}$$

Extra Notes for Section 2.20

Example 1: Evaluate $\displaystyle\int \frac{\sqrt{20-x^4}}{x}\,dx$.

Solution: It looks as though the substitution $u = x^2$ will put this integral in a form that appears in our tables. As a first step, we do some algebra so that du also shows up in the integral.

$$\begin{aligned}
\int \frac{\sqrt{20-x^4}}{x}\,dx &= \frac{1}{2}\int \frac{\sqrt{20-x^4}}{x^2}\,(2x\,dx) && \text{simple algebra}\\
&= \frac{1}{2}\int \frac{\sqrt{20-u^2}}{u}\,du && \text{let } u = x^2\text{, then } du = 2x\,dx\\
&= \frac{1}{2}\Big(\sqrt{20-u^2} - \sqrt{20}\,\ln\Big|\frac{\sqrt{20}+\sqrt{20-u^2}}{u}\Big|\Big) + C && \text{integral formula 19}\\
&= \frac{1}{2}\,\sqrt{20-x^4} - \sqrt{5}\,\ln\Big|\frac{\sqrt{20}+\sqrt{20-x^4}}{x^2}\Big| + C && \text{return to } x
\end{aligned}$$

Example 2: Evaluate $\displaystyle\int \frac{e^{3x}}{\sqrt{e^{2x}+25}}\,dx$.

Solution: In this case, it looks as though the substitution $u = e^x$ will put the integral in a form that appears in our tables. Once again, we begin with some algebra so that du shows up in the integral.

$$\begin{aligned}
\int \frac{e^{3x}}{\sqrt{e^{2x}+25}}\,dx &= \int \frac{e^{2x}}{\sqrt{e^{2x}+25}}\,(e^x\,dx) && \text{simple algebra}\\
&= \int \frac{u^2}{\sqrt{25+u^2}}\,du && \text{let } u = e^x\text{, then } du = e^x\,dx\\
&= \frac{u}{2}\,\sqrt{25+u^2} - \frac{25}{2}\,\ln\big|u+\sqrt{25+u^2}\,\big| + C && \text{integral formula 33}\\
&= \frac{e^x}{2}\,\sqrt{25+e^{2x}} - \frac{25}{2}\,\ln\big|e^x+\sqrt{25+e^{2x}}\,\big| + C && \text{return to } x
\end{aligned}$$

Example 3: Evaluate $\displaystyle\int \cos^5 x\,dx$.

Solution: We use the reduction formula for cosine (integral formula 63) multiple times.

$$\begin{aligned}
\int \cos^5 x\,dx &= \frac{1}{5}\,\cos^4 x\sin x + \frac{4}{5}\int \cos^3 x\,dx && \text{using } n = 5\\
&= \frac{1}{5}\,\cos^4 x\sin x + \frac{4}{5}\Big(\frac{1}{3}\,\cos^2 x\sin x + \frac{2}{3}\int \cos x\,dx\Big) && \text{using } n = 3\\
&= \frac{1}{5}\,\cos^4 x\sin x + \frac{4}{15}\,\cos^2 x\sin x + \frac{8}{15}\,\sin x + C && \text{basic formula}\\
&= \frac{1}{15}\,\sin x\big(3\cos^4 x + 4\cos^2 x + 8\big) + C && \text{if desired}
\end{aligned}$$

By reducing the power on cosine, we eventually reach an integral that we can evaluate directly.

Example 4: Evaluate $\int x\cos^2(2x^2)\,dx$.

Solution: In this case, we first make a substitution and then use the reduction formula for cosine.

$$\begin{aligned}
\int x\cos^2(2x^2)\,dx &= \frac{1}{4}\int \cos^2 u\,du && \text{let } u = 2x^2\\
&= \frac{1}{4}\Big(\frac{1}{2}\cos u\sin u + \frac{1}{2}\int du\Big) && \text{using } n = 2\\
&= \frac{1}{4}\Big(\frac{1}{2}\cos u\sin u + \frac{1}{2}u\Big) && \text{basic formula}\\
&= \frac{1}{8}\big(\cos(2x^2)\sin(2x^2) + 2x^2\big) + C && \text{return to } x
\end{aligned}$$

Example 5: Derive the reduction formula for cosine.

Solution: We proceed with integration by parts.

$$\int \cos^n x\,dx, \qquad \begin{array}{llll} \text{let} & u = \cos^{n-1}x & \text{and} & dv = \cos x\,dx;\\ \text{then} & du = (n-1)\cos^{n-2}x(-\sin x)\,dx & \text{and} & v = \sin x; \end{array}$$

We split things up in this way because using $dv = dx$ gives us $v = x$, making the new integral a product of a trig function and x which is more complicated. We now have

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1}x\sin x - \int (n-1)\cos^{n-2}x(-\sin^2 x)\,dx\\
\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx
\end{aligned}$$

At this stage, it looks as though we have made the integral more complicated. However, we can now use the basic trigonometric identity $\sin^2 x + \cos^2 x = 1$ to rewrite the integrand.

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx\\
\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx
\end{aligned}$$

We now think of $\int \cos^n x\,dx$ as our unknown, call it Y. The above equation is thus of the form

$$Y = W - (n-1)Y \quad \text{and thus} \quad Y + (n-1)Y = W \quad \Rightarrow \quad nY = W \quad \Rightarrow \quad Y = \frac{W}{n}$$

Carrying out these steps in full detail gives

$$\begin{aligned}
\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx\\
n\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx\\
\int \cos^n x\,dx &= \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx
\end{aligned}$$

This last equation is the reduction formula for cosine.

Extra Notes for Section 2.21

Example 1: Evaluate $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$.

Solution: It is possible to evaluate this integral with either a substitution ($u = 4 - x^2$) or integration by parts ($u = x^2$). However, we will use trigonometric substitution.

Let $x = 2\sin\theta$, then note that $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$.

It follows that

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\sin^3\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\
&= 8\int \sin^3\theta\,d\theta \\
&= 8\int \sin^2\theta\sin\theta\,d\theta \\
&= 8\int (1-\cos^2\theta)\sin\theta\,d\theta \\
&= 8\Big(\int \sin\theta\,d\theta + \int \cos^2\theta(-\sin\theta)\,d\theta\Big) \\
&= 8\Big(-\cos\theta + \frac{1}{3}\,\cos^3\theta\Big) + C \\
&= 8\Big(-\frac{\sqrt{4-x^2}}{2} + \frac{1}{3}\,\Big(\frac{\sqrt{4-x^2}}{2}\Big)^3\Big) + C \\
&= -4\sqrt{4-x^2} + \frac{1}{3}\,\big(4-x^2\big)^{3/2} + C \\
&= \frac{\sqrt{4-x^2}}{3}\big(-12+4-x^2\big) + C \\
&= -\frac{1}{3}(x^2+8)\sqrt{4-x^2} + C.
\end{aligned}$$

It would have been possible to use the reduction formula for sine at the third step; the above method presents another option. In addition, it is not necessary to do the last two steps as the equation that starts with -4 is a perfectly valid answer.

Example 2: Evaluate $\displaystyle\int \frac{x^2}{(x^2-9)^{3/2}}\,dx$.

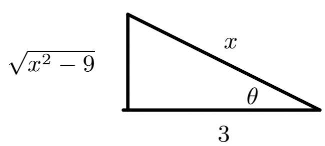
Solution: We will use trigonometric substitution for this integral.

Let $x = 3\sec\theta$, then note that $dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$.

It follows that

$$\begin{aligned}
\int \frac{x^2}{(x^2-9)^{3/2}}\,dx &= \int \frac{9\sec^2\theta}{27\tan^3\theta}\cdot 3\sec\theta\tan\theta\,d\theta \\
&= \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta \\
&= \int \frac{1}{\sin^2\theta\cos\theta}\,d\theta \\
&= \int \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos\theta}\,d\theta \\
&= \int \left(\sec\theta + \frac{\cos\theta}{\sin^2\theta}\right)d\theta \\
&= \ln\bigl|\sec\theta+\tan\theta\bigr| - \frac{1}{\sin\theta} + C.
\end{aligned}$$

To express these trig functions as functions of x, use the fact that $\sec\theta = x/3$ and form a right triangle containing the angle θ:



$$\sec\theta = \frac{x}{3};\qquad \sin\theta = \frac{\sqrt{x^2-9}}{x};\qquad \tan\theta = \frac{\sqrt{x^2-9}}{3}.$$

It follows that

$$\int \frac{x^2}{(x^2-9)^{3/2}}\,dx = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| - \frac{x}{\sqrt{x^2-9}} + C = \ln\bigl|x+\sqrt{x^2-9}\bigr| - \frac{x}{\sqrt{x^2-9}} + C.$$

Note that the second form of the solution just puts the term $-\ln 3$ from the logarithm into the constant C.

Extra Notes for Section 2.22

Example 1: Evaluate $\displaystyle\int \frac{3x-7}{x^2+5x+4}\,dx$.

Solution: Since the denominator can be factored, we can use partial fractions. The integrand is

$$\frac{3x-7}{x^2+5x+4} = \frac{3x-7}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1},$$

where A and B are constants. Multiplying through by the denominator $(x+4)(x+1)$ gives us

$$3x-7 = A(x+1) + B(x+4)$$

Since this equation must be true for all values of x, we can substitute any values we want for x. In this case, choosing values of x that conveniently make some of the terms equal 0, we find that

$$x=-4 \;\Rightarrow\; -19=-3A \;\Rightarrow\; A=\frac{19}{3}; \qquad \text{and} \qquad x=-1 \;\Rightarrow\; -10=3B \;\Rightarrow\; B=-\frac{10}{3}.$$

This technique can always be used when the denominator factors into distinct linear factors. Therefore,

$$\int \frac{3x-7}{x^2+5x+4}\,dx = \int \Big(\frac{19/3}{x+4} - \frac{10/3}{x+1}\Big)\,dx = \frac{19}{3}\,\ln|x+4| - \frac{10}{3}\,\ln|x+1| + C.$$

(To avoid a common error, be certain you place the constants you find over the appropriate factors.)

For a second method to find the constants A and B, we return to the second displayed equation and use the distributive property to expand the right-hand side, then combine like terms to obtain

$$3x-7 = A(x+1)+B(x+4) = Ax+A+Bx+4B = (A+B)x+(A+4B).$$

The only way for the two polynomials $3x-7$ and $(A+B)x+(A+4B)$ to be equal for all values of x is when $A+B=3$ and $A+4B=-7$, that is, the two polynomials must have the same coefficients. We must then solve two equations with two unknowns; you should be familiar with several ways to do this. In this case, we can subtract the two equations:

$$\begin{matrix} A+B=3 \\ A+4B=-7 \end{matrix} \quad\Rightarrow\quad -3B=10 \quad\Rightarrow\quad \begin{matrix} B=-10/3 \\ A=3-B=19/3. \end{matrix}$$

Of course, these are the same values found using the other method. This method of finding the values for the constants, namely matching up the coefficients of the polynomials, always works but it is sometimes more labor intensive than the first method. However, when irreducible quadratics are involved, this method may be the only reasonable choice.

Example 2: Evaluate $\displaystyle\int \frac{10x^2 - 5x + 3}{x^3 + x^2 - x + 15}\,dx$.

Solution: It is not essential that you remember how to factor cubics, but here is a brief refresher. The rational root theorem states that any rational roots of a polynomial must be of the form (factors of the constant term)/(factors of the leading coefficient). Thus for the polynomial $x^3 + x^2 - x + 15$, the possible rational roots are factors of 15 over factors of 1. We then check to see if any of the numbers $\pm 1, \pm 3, \pm 5, \pm 15$ are roots of the polynomial. Some trial and error reveals that -3 is a root and thus $x + 3$ is a factor. We then use long division to find that $x^3 + x^2 - x + 15 = (x + 3)(x^2 - 2x + 5)$. We can then check to see if the remaining quadratic can be factored. In this case, the quadratic $x^2 - 2x + 5$ cannot be factored; it is an irreducible quadratic. It follows that

$$\frac{10x^2 - 5x + 3}{x^3 + x^2 - x + 15} = \frac{10x^2 - 5x + 3}{(x + 3)(x^2 - 2x + 5)} = \frac{A}{x + 3} + \frac{Bx + C}{x^2 - 2x + 5}$$

for constants A, B, and C. Note that a linear term is needed over the irreducible quadratic. Multiplying through by the common denominator yields

$$\begin{aligned} 10x^2 - 5x + 3 &= A(x^2 - 2x + 5) + (Bx + C)(x + 3) \\ &= (Ax^2 - 2Ax + 5A) + (Bx^2 + 3Bx + Cx + 3C) \\ &= (A + B)x^2 + (-2A + 3B + C)x + (5A + 3C). \end{aligned}$$

Equating coefficients, we find that $A + B = 10$, $-2A + 3B + C = -5$, and $5A + 3C = 3$. To solve this system of equations (using one of many possible options), we begin by subtracting the third equation from 3 times the second equation to reduce the problem to two equations in two unknowns.

$$\begin{array}{c} A + B = 10 \\ -2A + 3B + C = -5 \\ 5A + 3C = 3 \end{array} \quad\Rightarrow\quad \begin{array}{c} A + B = 10 \\ -11A + 9B = -18 \end{array} \quad\Rightarrow\quad \begin{array}{c} 9A + 9B = 90 \\ -11A + 9B = -18 \end{array} \quad\Rightarrow\quad \begin{array}{c} 20A = 108 \\ A = \dfrac{27}{5} \end{array}$$

Once we know $A = 27/5$, we can use $A + B = 10$ to find $B = 23/5$ and $5A + 3C = 3$ to find $C = -8$. It then follows that

$$\begin{aligned} \int \frac{10x^2 - 5x + 3}{x^3 + x^2 - x + 15}\,dx &= \int \Big(\frac{27/5}{x + 3} + \frac{(23/5)x - 8}{x^2 - 2x + 5}\Big)\,dx \\ &= \int \Big(\frac{27/5}{x + 3} + \frac{(23/5)(x - 1) - (17/5)}{(x - 1)^2 + 4}\Big)\,dx \\ &= \frac{27}{5}\int \frac{1}{x + 3}\,dx + \frac{23}{5}\int \frac{x - 1}{(x - 1)^2 + 4}\,dx - \frac{17}{5}\int \frac{1}{(x - 1)^2 + 4}\,dx \\ &= \frac{27}{5}\ln|x + 3| + \frac{23}{10}\ln\big|(x - 1)^2 + 4\big| - \frac{17}{10}\arctan\Big(\frac{x - 1}{2}\Big) + C. \end{aligned}$$

If this algebraic approach for evaluating the integral is confusing to you, let $u = x - 1$ in the integral involving the irreducible quadratic and proceed as we did in Section 2.19.

Extra Notes for Section 2.23

Example 1: Use both the trapezoid rule and Simpson's rule with $n = 8$ to approximate $\int_0^2 \frac{1-\cos x}{x^2}\,dx$. Give your answers correct to four decimal places.

Solution: Let $f(x) = (1 - \cos x)/x^2$. Since $(2-0)/8 = 0.25$, the formulas for the trapezoid rule and Simpson's rule yield

$$T_8 = \frac{2}{2\cdot 8}\big(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)\big);$$
$$S_8 = \frac{2}{3\cdot 8}\big(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)\big);$$

respectively. Using a calculator, we find that (note that we keep extra decimal places at this stage in order to avoid round-off error)

$$f(0) = 0.5; \qquad f(0.75) \approx 0.476998; \qquad f(1.5) \approx 0.413006;$$
$$f(0.25) \approx 0.497401; \qquad f(1.0) \approx 0.459698; \qquad f(1.75) \approx 0.384733;$$
$$f(0.5) \approx 0.489670; \qquad f(1.25) \approx 0.438194; \qquad f(2.0) \approx 0.354037.$$

The value for $f(0)$ is determined by

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2},$$

where we have used L'Hôpital's Rule to help evaluate the limit. Adding the appropriate numbers (as indicated by the equations for T_8 and S_8), we obtain

0.500000	0.500000
0.994802	1.989604
0.979340	0.979340
0.953996	1.907992
0.919396	0.919396
0.876388	1.752776
0.826012	0.826012
0.769466	1.538932
0.354037	0.354037
7.173437	10.768089

It then follows that $T_8 \approx 7.173437/8 \approx 0.8967$ and $S_8 \approx 10.768089/12 \approx 0.8973$, accurate to four decimal places. For the record, the actual value of the integral is approximately 0.89734.

Example 2: Consider the following table of values for a function f:

x	0	1	2	3	4	5	6
$f(x)$	1	2	3	5	7	2	4

Use this information and Simpson's rule to approximate $\int_0^6 x\big(f(x)\big)^2\,dx$.

Solution: Given the table of values, it is clear that we should use $n = 6$. Using the formula for Simpson's rule (with the integrand $x(f(x))^2$) yields the following value for S_6:

$$\begin{aligned}
&\frac{6}{3\cdot 6}\Big(\big(0\cdot f(0)^2\big) + 4\big(1\cdot f(1)^2\big)\big) + 2\big(2\cdot f(2)^2\big) + 4\big(3\cdot f(3)^2\big) + 2\big(4\cdot f(4)^2\big) + 4\big(5\cdot f(5)^2\big) + \big(6\cdot f(6)^2\big)\Big)\\
&\qquad = \frac{1}{3}\Big(0 + 4\cdot 2^2 + 4\cdot 3^2 + 12\cdot 5^2 + 8\cdot 7^2 + 20\cdot 2^2 + 6\cdot 4^2\Big)\\
&\qquad = \frac{1}{3}\big(16 + 36 + 300 + 392 + 80 + 96\big)\\
&\qquad = \frac{1}{3}\cdot 920.
\end{aligned}$$

It then follows that $\int_0^6 x\big(f(x)\big)^2\,dx \approx \dfrac{920}{3}$.