Extra Notes for Section 3.1

Prior to studying these examples, you should read the material in the latter part of the Prelude to Chapter 3 in order to become more comfortable with the notation and basic ideas.

Example 1: For each positive integer n, the number $\frac{1}{5}\,n^5 + \frac{1}{3}\,n^3 + \frac{7}{15}\,n$ is an integer.

Solution: We will use the Principle of Mathematical Induction. We first check the statement for the first few values of n.

$$\begin{aligned}
n = 1: &\qquad \frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{3+5+7}{15} = 1;\\
n = 2: &\qquad \frac{32}{5} + \frac{8}{3} + \frac{14}{15} = \frac{96+40+14}{15} = 10;\\
n = 3: &\qquad \frac{243}{5} + 9 + \frac{7}{5} = 48 + \frac{3}{5} + 9 + \frac{7}{5} = 59.
\end{aligned}$$

We now know that the expression gives an integer when n is 1, 2, or 3. Suppose that $\frac{1}{5}\,k^5 + \frac{1}{3}\,k^3 + \frac{7}{15}\,k$ is an integer for some positive integer k; let's call this integer m. Using the appropriate rows of Pascal's triangle, we obtain

$$\begin{aligned}
\frac{1}{5}\,(k+1)^5 + &\frac{1}{3}\,(k+1)^3 + \frac{7}{15}\,(k+1)\\
&= \frac{1}{5}\big(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\big) + \frac{1}{3}\big(k^3 + 3k^2 + 3k + 1\big) + \frac{7}{15}\,(k+1)\\
&= \Big(\frac{1}{5}\,k^5 + \frac{1}{3}\,k^3 + \frac{7}{15}\,k\Big) + (k^4 + 2k^3 + 2k^2 + k) + (k^2 + k) + \Big(\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\Big)\\
&= m + k^4 + 2k^3 + 3k^2 + 2k + 1,
\end{aligned}$$

which shows that the expression is an integer when $n = k+1$. By the Principle of Mathematical Induction, the number $\frac{1}{5}\,n^5 + \frac{1}{3}\,n^3 + \frac{7}{15}\,n$ is an integer for each positive integer n.

Example 2: Suppose that $a_0 = 3$, $a_1 = 7$, and $a_{n+1} = 5a_n - 6a_{n-1}$ for each positive integer $n \ge 1$. Find and prove a simple expression for a_n.

Solution: Gathering some data, we find that

$$a_2 = 5a_1 - 6a_0 = 17, \quad a_3 = 5a_2 - 6a_1 = 43, \quad a_4 = 5a_3 - 6a_2 = 113.$$

After spending some time looking for patterns in the numbers 3, 7, 17, 43, and 113, we discover that

$$3 = 2 + 1 \Leftrightarrow a_0 = 2^1 + 3^0, \quad 7 = 4 + 3, \quad 17 = 8 + 9, \quad 43 = 16 + 27, \quad 113 = 32 + 81 \Leftrightarrow a_4 = 2^5 + 3^4.$$

It appears that $a_n = 2^{n+1} + 3^n$ for each nonnegative integer n. We can prove this using the Principle of Strong Induction. We already know that the expression for a_n is valid when n is 0, 1, 2, 3, or 4. Suppose that $a_i = 2^{i+1} + 3^i$ for all integers $i = 0, 1, 2, \ldots, k$ for some positive integer k. Then (follow the steps closely)

$$a_{k+1} = 5a_k - 6a_{k-1} = 5(2^{k+1} + 3^k) - 6(2^k + 3^{k-1}) = (10 - 6)\cdot 2^k + (15 - 6)\cdot 3^{k-1} = 2^{k+2} + 3^{k+1}.$$

This shows that the expression for a_{k+1} is also valid. By the Principle of Strong Induction, we have shown that $a_n = 2^{n+1} + 3^n$ for each nonnegative integer n.

Example 3: For each positive integer n, $\sum_{i=1}^{n} if_i = nf_{n+2} - f_{n+3} + 2$.

Proof: We will use the Principle of Mathematical Induction. We first check the equation for a few values of n (assuming the reader is familiar with the Fibonacci sequence). For $n = 1$, we have

$$\sum_{i=1}^{1} if_i = f_1 = 1 \quad \text{and} \quad 1 \cdot f_3 - f_4 + 2 = 1 \cdot 2 - 3 + 2 = 1.$$

For $n = 2$, we have

$$\sum_{i=1}^{2} if_i = f_1 + 2f_2 = 1 + 2 \cdot 1 = 3 \quad \text{and} \quad 2 \cdot f_4 - f_5 + 2 = 2 \cdot 3 - 5 + 2 = 3.$$

For $n = 3$, we have

$$\sum_{i=1}^{3} if_i = f_1 + 2f_2 + 3f_3 = 1 + 2 \cdot 1 + 3 \cdot 2 = 9 \quad \text{and} \quad 3 \cdot f_5 - f_6 + 2 = 3 \cdot 5 - 8 + 2 = 9.$$

We now know that the equation is valid for $n = 1, 2, 3$. Suppose that $\sum_{i=1}^{k} if_i = kf_{k+2} - f_{k+3} + 2$ for some positive integer k. We then have

$$\begin{aligned}
\sum_{i=1}^{k+1} if_i &= \sum_{i=1}^{k} if_i + (k+1)f_{k+1} \\
&= \big(kf_{k+2} - f_{k+3} + 2\big) + (k+1)f_{k+1} \\
&= (k+1)f_{k+2} - f_{k+2} - f_{k+3} + 2 + (k+1)f_{k+1} \\
&= (k+1)\big(f_{k+1} + f_{k+2}\big) - \big(f_{k+2} + f_{k+3}\big) + 2 \\
&= (k+1)f_{k+3} - f_{k+4} + 2,
\end{aligned}$$

which is the desired equality for $n = k + 1$. By the Principle of Mathematical Induction, the equation $\sum_{i=1}^{n} if_i = nf_{n+2} - f_{n+3} + 2$ is valid for all positive integers n.

Example 4: Every positive integer $n \geq 2$ is either a prime number or can be factored into a product of prime numbers.

Solution: We will use the Principle of Strong Induction. It is clear that 2 is a prime number. Suppose that for some positive integer $k \geq 2$, each of the integers $2, 3, \ldots, k$ is either a prime number or can be factored into a product of prime numbers. Consider the integer $k + 1$. If $k + 1$ is a prime number, there is nothing further to prove. If $k + 1$ is not a prime number, then $k + 1 = ab$, where a and b are integers between 2 and k, inclusively. By the induction hypothesis, each of the integers a and b is either a prime number or can be factored into a product of prime numbers. It follows that $k + 1 = ab$ can be factored into a product of prime numbers. By the Principle of Strong Induction, every integer $n \geq 2$ is either a prime number or can be factored into a product of prime numbers.

Consider the statement $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n \cdot (3n+1) = n(n+1)^2$. To claim that this statement is true for every positive integer n means that all of the following equations are true.

$$
\begin{aligned}
1 \cdot 4 &= 1 \cdot 2^2; \\
1 \cdot 4 + 2 \cdot 7 &= 2 \cdot 3^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 &= 3 \cdot 4^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 &= 4 \cdot 5^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 &= 5 \cdot 6^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 + 6 \cdot 19 &= 6 \cdot 7^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 + 6 \cdot 19 + 7 \cdot 22 &= 7 \cdot 8^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 + 6 \cdot 19 + 7 \cdot 22 + 8 \cdot 25 &= 8 \cdot 9^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 + 6 \cdot 19 + 7 \cdot 22 + 8 \cdot 25 + 9 \cdot 28 &= 9 \cdot 10^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + 5 \cdot 16 + 6 \cdot 19 + 7 \cdot 22 + 8 \cdot 25 + 9 \cdot 28 + 10 \cdot 31 &= 10 \cdot 11^2; \\
&\vdots \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + \cdots + k \cdot (3k+1) &= k(k+1)^2; \\
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + \cdots + k \cdot (3k+1) + (k+1) \cdot (3k+4) &= (k+1)(k+2)^2; \\
&\vdots
\end{aligned}
$$

It would not be all that difficult to check the first ten of these statements by hand. A computer could easily check the next million cases or so but the numbers would get rather large. However, it is impossible to check all of the equations since there are an infinite number of them. This is where the Principle of Mathematical Induction comes into play. In the set of equations above, the letter k is used to represent some generic integer that is larger than 10. If we can show that the validity of the k equation forces the validity of the $k+1$ equation, then we know that all of the equations are true; each correct equation forces the next one to be true as well and so the equations are true “all the way down”.

Here is an approximate template for writing solutions to induction problems. However, there will be some problems in which modifications to the template are needed.

1. We will use the Principle of Mathematical Induction (Principle of Strong Induction).
2. Let S be the set of all positive integers n such that ...
3. Since [do some simple math] ..., it follows that $1 \in S$.
4. Now suppose that $k \in S$ (suppose that $1, 2, \ldots, k \in S$) for some positive integer k.
5. Since $k \in S$, we know that [write what it means]
6. We then have [do some work here, some of it may be complicated] ...
7. It follows that $k+1 \in S$.
8. We have thus shown that the conditional statement “if $k \in S$, then $k+1 \in S$” (“if $1, 2, \ldots, k \in S$, then $k+1 \in S$” is true). By the Principle of Mathematical Induction (Principle of Strong Induction), we know that $S = \mathbb{Z}^+$. It follows that [repeat the statement that you are proving] for each $n \in \mathbb{Z}^+$.

Here is an example to illustrate the template approach to solving such problems. It is the statement highlighted on the previous page. The second solution is the one found more often in textbooks.

Problem: Prove that $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n \cdot (3n+1) = n(n+1)^2$ for each positive integer n.

Solution: We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n such that $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n \cdot (3n+1) = n(n+1)^2$. Since $1 \cdot 4 = 1 \cdot 2^2$, it is clear that $1 \in S$. Now suppose that $k \in S$ for some positive integer k. This means that

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + \cdots + k \cdot (3k+1) = k(k+1)^2.$$

We then have

$$\begin{aligned}
&1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + (k+1) \cdot (3(k+1)+1) && \text{left-hand side of } k+1 \text{ equation} \\
&\quad = 1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + k \cdot (3k+1) + (k+1) \cdot (3(k+1)+1) && \text{include a term} \\
&\quad = k(k+1)^2 + (k+1) \cdot (3(k+1)+1) && \text{using our hypothesis} \\
&\quad = (k+1)\big(k(k+1) + (3k+4)\big) && \text{factor out a common term} \\
&\quad = (k+1)\big(k^2 + 4k + 4\big) && \text{combine like terms} \\
&\quad = (k+1)(k+2)^2 && \text{recognize a perfect square} \\
&\quad = (k+1)\big((k+1)+1\big)^2, && \text{right-hand side of } k+1 \text{ equation}
\end{aligned}$$

which tells us that $k+1 \in S$. We have thus shown that the conditional statement "if $k \in S$, then $k+1 \in S$" is true. By the Principle of Mathematical Induction, we know that $S = \mathbb{Z}^+$. It follows that

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n \cdot (3n+1) = n(n+1)^2$$

for each positive integer n. ■

Abbreviated Solution: We will use the Principle of Mathematical Induction. Since $1 \cdot 4 = 1 \cdot 2^2$, the equation is valid when $n = 1$. Now suppose that

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + 4 \cdot 13 + \cdots + k \cdot (3k+1) = k(k+1)^2.$$

for some positive integer k. We then have

$$\begin{aligned}
1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + k \cdot (3k+1) + (k+1) \cdot (3k+4) &= k(k+1)^2 + (k+1) \cdot (3k+4) \\
&= (k+1)\big(k(k+1) + (3k+4)\big)) \\
&= (k+1)\big(k^2 + 4k + 4\big)) \\
&= (k+1)(k+2)^2,
\end{aligned}$$

revealing that the equation is valid for $n = k+1$. By the Principle of Mathematical Induction, it follows that

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n \cdot (3n+1) = n(n+1)^2$$

for each positive integer n. ■

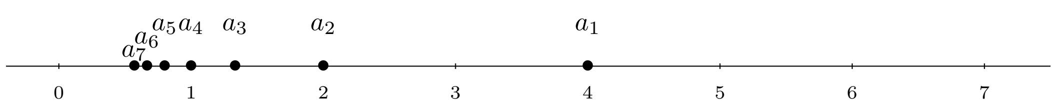
Extra Notes for Section 3.2

It is important to be familiar with many examples of sequences of real numbers. Note that there are many types of sequences; any list of numbers that goes on forever is a sequence.

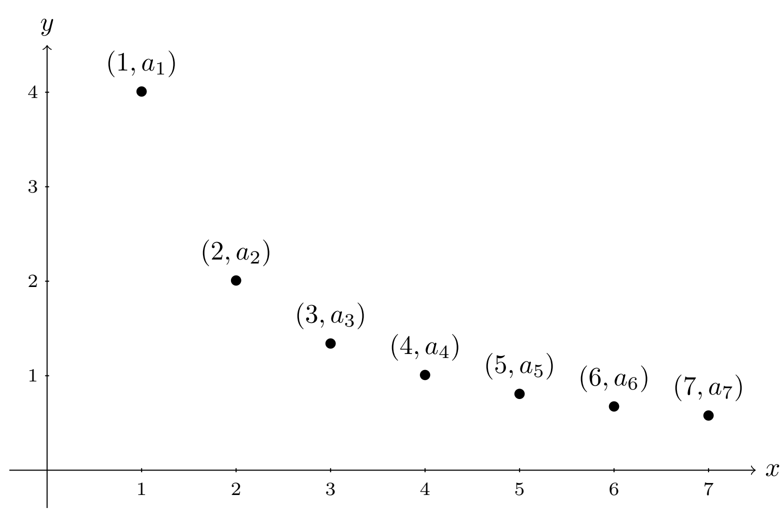
1. $\{2^n\} = 2, 4, 8, 16, 32, \ldots$
2. $\{3^{1-n}\} = 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
3. $\left\{\frac{(-1)^n}{n}\right\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$
4. $\{\sqrt{n}\} = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots$
5. $\{(-1)^{n+1}n\} = 1, -2, 3, -4, 5, \ldots$
6. $\{n^2\} = 1, 4, 9, 16, 25, \ldots$
7. $\left\{\frac{n}{3n-1}\right\} = \frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \frac{5}{14}, \ldots$
8. $\left\{\frac{1+(-1)^{n+1}}{2}\right\} = 1, 0, 1, 0, 1, \ldots$
9. $\{\sqrt[n]{n}\} = 1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \ldots$
10. $\left\{\left(1+\frac{6}{n}\right)^n\right\} = 7, 4^2, 3^3, \left(\frac{5}{2}\right)^4, \left(\frac{11}{5}\right)^5, \ldots$
11. $\left\{\frac{2^n}{1\cdot 4\cdot 7\cdot\ \cdots\ \cdot(3n+1)}\right\} = \frac{2}{1\cdot 4}, \frac{2^2}{1\cdot 4\cdot 7}, \frac{2^3}{1\cdot 4\cdot 7\cdot 10}, \frac{2^4}{1\cdot 4\cdot 7\cdot 10\cdot 13}, \frac{2^5}{1\cdot 4\cdot 7\cdot 10\cdot 13\cdot 16}, \ldots$
12. $\left\{\sum_{k=1}^{n}\frac{1}{k}\right\} = 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \ldots$

Study these examples carefully and be absolutely certain that you understand the terms that are generated by the various formulas. Which adjectives from Definition 3.4 apply to each of these sequences?

There are two ways to visualize sequences graphically; we illustrate these for the sequence $\{a_n\} = \{4/n\}$. For a one dimensional view, think of each term of the sequence as a dot on the number line; the dots are like stepping stones that must be traversed in a certain order: a_1, a_2, a_3, ...



For a two dimensional view, think of a_n as the value of a function $a(n)$, where the inputs are integers. We can then represent the graph of such a function as a bunch of dots in the plane.



It is very important that you be able to use these two representations of sequences to visualize the descriptive adjectives for sequences that appear in Definition 3.4.

Example 1: Determine the first five terms of the sequence $\left\{\sum_{k=0}^{n} \frac{1}{k!}\right\}$.

Solution: Substituting 1, 2, 3, 4, and 5 for n into the formula for the terms, we find that

$$\sum_{k=0}^{1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} = 1 + 1 = 2;$$
$$\sum_{k=0}^{2} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = 1 + 1 + \frac{1}{2} = \frac{5}{2};$$
$$\sum_{k=0}^{3} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3};$$
$$\sum_{k=0}^{4} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24};$$
$$\sum_{k=0}^{5} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60}.$$

The first five terms of the sequence are thus $2, \frac{5}{2}, \frac{8}{3}, \frac{65}{24}, \frac{163}{60}$.

Example 2: Find the limit of the sequence $\left\{\sqrt{n^2+5n} - n\right\}$.

Solution: We use the technique of multiplying by the conjugate. Since

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+5n} - n\right) &= \lim_{n\to\infty}\left(\left(\sqrt{n^2+5n} - n\right)\cdot\frac{\sqrt{n^2+5n}+n}{\sqrt{n^2+5n}+n}\right) \\
&= \lim_{n\to\infty}\frac{n^2+5n-n^2}{\sqrt{n^2+5n}+n} \\
&= \lim_{n\to\infty}\left(\frac{5n}{\sqrt{n^2+5n}+n}\cdot\frac{1/n}{1/n}\right) \\
&= \lim_{n\to\infty}\frac{5}{\sqrt{1+\frac{5}{n}}+1} = \frac{5}{2},
\end{aligned}$$

(note that $1/n$ entered the square root as $1/n^2$), the limit of the sequence $\left\{\sqrt{n^2+5n} - n\right\}$ is $5/2$.

Example 3: Without finding the sum, show that the 16th term of the sequence $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}$ is greater than 3.

Solution: We use some convenient pairing and estimation to make the sum easier to work with.

$$\begin{aligned}
\sum_{k=1}^{16} \frac{1}{k} &= 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right) \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} = 3.
\end{aligned}$$

It takes some practice to learn how to do estimates such as this but it is an important skill to acquire.

Extra Notes for Section 3.3

We begin with some examples to illustrate Theorem 3.8; study them carefully.

1. The sequences $\left\{\frac{1}{n^4}\right\}$, $\left\{\frac{1}{\sqrt{n}}\right\}$, and $\left\{\frac{1}{\sqrt[8]{n}}\right\}$ all converge to 0.
2. The sequences $\left\{\left(\frac{2}{3}\right)^n\right\}$, $\left\{\left(-\frac{6}{7}\right)^n\right\}$, and $\left\{\frac{1}{e^n}\right\}$ all converge to 0.
3. The sequences $\left\{n^4\left(-\frac{3}{4}\right)^n\right\}$, $\left\{\frac{n^5}{5^n}\right\}$, and $\left\{\frac{n^{10}}{1.2^n}\right\}$ all converge to 0.
4. The sequences $\left\{\left(\frac{7}{4}\right)^n\right\}$, $\left\{\left(-\frac{3}{2}\right)^n\right\}$, and $\left\{\pi^n\right\}$ are unbounded.
5. The sequences $\left\{\sqrt[n]{7}\right\}$, $\left\{\sqrt[n]{100}\right\}$, and $\left\{\sqrt[n]{2018}\right\}$ all converge to 1.
6. The sequences $\left\{\sqrt[n]{n}\right\}$, $\left\{\sqrt[k]{k}\right\}$, and $\left\{\sqrt[j]{j}\right\}$ all converge to 1.
7. The sequences $\left\{\left(1+\frac{2}{n}\right)^n\right\}$, $\left\{\left(1+\frac{1}{2n}\right)^n\right\}$, and $\left\{\left(1-\frac{1}{2n}\right)^n\right\}$ converge to e^2, $e^{1/2}$, and $e^{-1/2}$.

Combining these examples with Theorem 6, we can obtain the following results.

a. The sequences $\left\{\frac{12}{\sqrt[8]{n}}\right\}$ and $\left\{8\sqrt[n]{100}\right\}$ converge to 0 and 8, respectively.

b. The sequences $\left\{\sqrt[n]{7}+\sqrt[n]{n}\right\}$ and $\left\{\frac{n^5}{5^n}+\sqrt[n]{2018}\right\}$ converge to 2 and 1, respectively.

c. The sequences $\left\{\frac{1}{\sqrt{n}}-\sqrt[n]{n}\right\}$ and $\left\{\sqrt[n]{100}-\left(-\frac{6}{7}\right)^n\right\}$ converge to -1 and 1, respectively.

d. The sequences $\left\{\sqrt[n]{n}\left(1+\frac{2}{n}\right)^n\right\}$ and $\left\{\frac{1}{e^n}\cdot\left(1+\frac{1}{2n}\right)^n\right\}$ converge to e^2 and 0, respectively.

e. The sequences $\left\{\frac{\sqrt[n]{n}}{\sqrt[n]{7}}\right\}$ and $\left\{\frac{\sqrt[n]{100}}{\left(1-\frac{1}{2n}\right)^n}\right\}$ converge to 1 and $\sqrt{e}$, respectively.

f. The sequence $\left\{3\sqrt[n]{7n}-2\left(1+\frac{2}{n}\right)^n\right\}$ converges to $3-2e^2$.

Finally, to illustrate the Squeeze Theorem, consider the sequence $\left\{\sqrt[n]{5n^2+2n}\right\}$. Note that

$$2n \le 5n^2+2n \le 7n^2 \quad \text{and thus} \quad \sqrt[n]{2n} \le \sqrt[n]{5n^2+2n} \le \sqrt[n]{7n^2}$$

for all positive integers n. Since the sequences $\left\{\sqrt[n]{2n}\right\}$ and $\left\{\sqrt[n]{7n^2}\right\}$ both converge to 1, the sequence $\left\{\sqrt[n]{5n^2+2n}\right\}$ must also converge to 1. (If you are uncertain about the limits of the two new sequences, note that they can be written as $\left\{\sqrt[n]{2}\cdot\sqrt[n]{n}\right\}$ and $\left\{\sqrt[n]{7}\cdot\sqrt[n]{n}\cdot\sqrt[n]{n}\right\}$, respectively.)

Although Exercise 1 in the textbook asks you to write out a detailed proof for the limit of a sequence (in order to help you better understand the basic properties of convergent sequences), usually we do not go into this much detail. The following examples illustrate this.

Example 1: Find the limit of the sequence $\left\{\dfrac{2^n+n^5}{3^n+n}\right\}$.

Solution: Using parts (2) and (3) of Theorem 3.8, we find that

$$\lim_{n\to\infty}\frac{2^n+n^5}{3^n+n}=\lim_{n\to\infty}\left(\frac{2^n+n^5}{3^n+n}\cdot\frac{1/3^n}{1/3^n}\right)=\lim_{n\to\infty}\frac{\left(\frac{2}{3}\right)^n+\frac{n^5}{3^n}}{1+\frac{n}{3^n}}=\frac{0+0}{1+0}=0.$$

The limit of the sequence $\left\{\dfrac{2^n+n^5}{3^n+n}\right\}$ is thus 0. We chose the multiplier $1/3^n$ because 3^n is the term in the denominator that grows most quickly.

Example 2: Find the limit of the sequence $\left\{\dfrac{3^{2n-1}+4^n n}{7^{n+1}+9^{n-1}}\right\}$.

Solution: Using parts (2) and (3) of Theorem 3.8, we find that

$$\lim_{n\to\infty}\frac{3^{2n-1}+4^n n}{7^{n+1}+9^{n-1}}=\lim_{n\to\infty}\left(\frac{3^{2n}\cdot 3^{-1}+4^n n}{7\cdot 7^n+9^{-1}\cdot 9^n}\cdot\frac{1/9^n}{1/9^n}\right)=\lim_{n\to\infty}\frac{\frac{1}{3}+n\left(\frac{4}{9}\right)^n}{7\cdot\left(\frac{7}{9}\right)^n+\frac{1}{9}}=\frac{\frac{1}{3}+0}{0+\frac{1}{9}}=3.$$

Therefore, the limit of the sequence $\left\{\dfrac{3^{2n-1}+4^n n}{7^{n+1}+9^{n-1}}\right\}$ is 3. (We multiplied numerator and denominator by $1/9^n$ since it is the term in the denominator that grows most rapidly.)

Example 3: Find the limit of the sequence $\left\{\sqrt[n]{3^n+\pi^n}\right\}$.

Solution: The idea here is that for really large values of n, the number π^n will be much bigger than 3^n (since $\pi>3$). Ignoring the number 3^n, we end up with the nth root of π^n, which is just π. It thus appears that the limit of the sequence will be π. To prove this, we use the Squeeze Theorem. Note that

$$\pi^n\le 3^n+\pi^n\le 2\pi^n\quad\text{and thus}\quad\pi\le\sqrt[n]{3^n+\pi^n}\le\pi\sqrt[n]{2}$$

for all positive integers n. Since the sequences $\{\pi\}$ (a constant sequence) and $\{\pi\sqrt[n]{2}\}$ both converge to π (using part (6) of Theorem 3.8), the sequence $\left\{\sqrt[n]{3^n+\pi^n}\right\}$ must also converge to π by the Squeeze Theorem.

Example 4: Find the limit of the sequence $\left\{\left(\dfrac{3n}{3n+1}\right)^n\right\}$.

Solution: Using some basic algebra, we find that

$$\left(\frac{3n}{3n+1}\right)^n=\left(\frac{3n+1}{3n}\right)^{-n}=\left(\left(1+\frac{1/3}{n}\right)^n\right)^{-1}.$$

By part (7) of Theorem 3.8, the sequence $\left\{\left(1+\dfrac{1/3}{n}\right)^n\right\}$ converges to $e^{1/3}$. It follows that (see part (e) of Theorem 3.6) the sequence $\left\{\left(\dfrac{3n}{3n+1}\right)^n\right\}$ converges to $e^{-1/3}$.

Extra Notes for Section 3.4

Example 1: Consider the sequence $\{a_n\}$ defined by $a_1 = 2$ and $a_{n+1} = 5 - \dfrac{2}{a_n}$ for each positive integer n. Prove that $\{a_n\}$ converges and find its limit.

Solution: Using the recursive definition of the sequence, we find that

$$a_2 = 5 - \frac{2}{a_1} = 5 - \frac{2}{2} = 4; \qquad a_4 = 5 - \frac{2}{a_3} = 5 - \frac{2}{9/2} = \frac{41}{9} \approx 4.5556;$$

$$a_3 = 5 - \frac{2}{a_2} = 5 - \frac{2}{4} = \frac{9}{2} = 4.5; \qquad a_5 = 5 - \frac{2}{a_4} = 5 - \frac{2}{41/9} = \frac{187}{41} \approx 4.5610.$$

From these first few terms of the sequence, we conjecture that the sequence $\{a_n\}$ is increasing and bounded by 5. To verify this, we will prove the following two statements using mathematical induction:

I. the inequality $2 \le a_n < 5$ is valid for each positive integer n;

II. the inequality $a_n < a_{n+1}$ is valid for each positive integer n.

To prove I, we already know that the inequality is true for $n = 1$ so assume that $2 \le a_k < 5$ for some positive integer k. To determine the next term a_{k+1} in the sequence, we refer to the recursive definition of the sequence and see that we need to take the reciprocal of a_k, multiply it by -2, and then add 5. Performing these steps (and being careful with the inequalities) yields

$$2 \le a_k < 5 \quad \Rightarrow \quad \frac{1}{2} \ge \frac{1}{a_k} > \frac{1}{5} \quad \Rightarrow \quad -1 \le -\frac{2}{a_k} < -\frac{2}{5} \quad \Rightarrow \quad 4 \le 5 - \frac{2}{a_k} < 5 - \frac{2}{5} \quad \Rightarrow \quad 4 \le a_{k+1} < 4.6.$$

It then follows that $2 \le a_{k+1} < 5$. By the Principle of Mathematical Induction, the inequality $2 \le a_n < 5$ is valid for each positive integer n. We conclude that the sequence $\{a_n\}$ is bounded.

To prove II, we once again know that the inequality is true for $n = 1$ since $a_1 < a_2$. Assume that $a_k < a_{k+1}$ for some positive integer k. Proceeding in much the same way as in the previous paragraph, we find that (for the record, we are using the fact that all of the terms of the sequence are positive)

$$a_k < a_{k+1} \quad \Rightarrow \quad \frac{1}{a_k} > \frac{1}{a_{k+1}} \quad \Rightarrow \quad -\frac{2}{a_k} < -\frac{2}{a_{k+1}} \quad \Rightarrow \quad 5 - \frac{2}{a_k} < 5 - \frac{2}{a_{k+1}} \quad \Rightarrow \quad a_{k+1} < a_{k+2},$$

which is the appropriate inequality for $n = k+1$. By the Principle of Mathematical Induction, the inequality $a_n < a_{n+1}$ is valid for each positive integer n. It follows that the sequence $\{a_n\}$ is increasing.

Since the sequence $\{a_n\}$ is bounded and monotone, it converges by the Completeness Axiom. Let L be the limit of the sequence. As n goes to infinity, the recurrence relation

$$a_{n+1} = 5 - \frac{2}{a_n} \quad \text{becomes} \quad L = 5 - \frac{2}{L}$$

since the a_n terms approach L as n gets larger. Solving for L (using the quadratic formula) gives

$$L^2 = 5L - 2 \quad \Rightarrow \quad L^2 - 5L + 2 = 0 \quad \Rightarrow \quad L = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}.$$

Since the sequence $\{a_n\}$ is increasing, we know that the limit has to be larger than 2 so the L value corresponding to the minus sign is too small. Hence, the limit of the sequence $\{a_n\}$ is $\dfrac{5+\sqrt{17}}{2} \approx 4.5615528$.

Example 2: Prove that the sequence $\left\{\sum_{k=1}^{n} \frac{1}{k2^k}\right\}$ converges.

Solution: For each positive integer n, let $b_n = \sum_{k=1}^{n} \frac{1}{k2^k}$; we want to show that the sequence $\{b_n\}$ converges. Using the formula for the terms, we find that the first few terms of this sequence are

$$b_1 = \sum_{k=1}^{1} \frac{1}{k2^k} = \frac{1}{1 \cdot 2^1};$$

$$b_2 = \sum_{k=1}^{2} \frac{1}{k2^k} = \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2};$$

$$b_3 = \sum_{k=1}^{3} \frac{1}{k2^k} = \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3};$$

$$b_4 = \sum_{k=1}^{4} \frac{1}{k2^k} = \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4};$$

$$b_5 = \sum_{k=1}^{5} \frac{1}{k2^k} = \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5};$$

$$b_6 = b_5 + \frac{1}{6 \cdot 2^6}.$$

It should be obvious that the sequence $\{b_n\}$ is increasing. (It is very important that you see this; you get b_{n+1} from b_n by adding a positive number to b_n.) To prove that $\{b_n\}$ converges, we need to prove that it is bounded. Using some simple overestimates (when the denominator of a fraction is smaller the fraction is larger) and the formula from Exercise 7 in Section 2.1, we obtain

$$\begin{aligned} b_n = \sum_{k=1}^{n} \frac{1}{k2^k} &= \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{n \cdot 2^n} \\ &\leq \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1}{2} \cdot \frac{(1/2)^n - 1}{(1/2) - 1} = 1 - \left(\frac{1}{2}\right)^n < 1 \end{aligned}$$

for each positive integer n. This shows that the sequence $\{b_n\}$ is bounded by 1. Since the sequence $\{b_n\}$ is bounded and monotone, it converges by the Completeness Axiom. For the record, the limit of this sequence is not 1; see Example 5 in the next set of notes.

For ease of reference, we record five ways to prove that a sequence $\{a_n\}$ is increasing; there are analogous ways to prove that a sequence is decreasing.

1. Show algebraically that $a_{n+1} \geq a_n$ for each n.
2. Show algebraically that $a_{n+1} - a_n \geq 0$ for each n.
3. Show algebraically that $a_{n+1}/a_n \geq 1$ for each n, ASSUMING all the a_n terms are positive.
4. Show that $a_{n+1} \geq a_n$ for each n using the Principle of Mathematical Induction.
5. Convert the formula for a_n to a function $f(x)$ and show that $f'(x) \geq 0$ for all $x \geq 1$.

Often the form of the sequence will provide an indication of which of these five approaches is best for a particular sequence.

Extra Notes for Section 3.5

Example 1: Find an explicit formula for the sequence of partial sums for the series $\sum_{k=1}^{\infty} \frac{6}{4^k}$.

Solution: Using the formula from Exercise 7 in Section 2.1, we find that the nth partial sum of this series is

$$\sum_{k=1}^{n} \frac{6}{4^k} = 6\sum_{k=1}^{n}\Big(\frac{1}{4}\Big)^k = 6 \cdot \frac{1}{4} \cdot \frac{1-(1/4)^n}{1-(1/4)} = 2\Big(1 - \frac{1}{4^n}\Big).$$

Example 2: Prove that the series $\sum_{k=1}^{\infty} \frac{5}{3+\sqrt[k]{k^2}}$ diverges.

Solution: We will use the Divergence Test. Referring to a previous result (part (6) of Theorem 3.8), we find that

$$\lim_{k\to\infty} \frac{5}{3+\sqrt[k]{k^2}} = \lim_{k\to\infty} \frac{5}{3+\sqrt[k]{k}\cdot\sqrt[k]{k}} = \frac{5}{3+1\cdot 1} = \frac{5}{4}.$$

Since the terms of the series do not converge to 0, the series diverges by the Divergence Test.

Example 3: Find the sum of the series $\sum_{k=2}^{\infty} \frac{3^k}{2^{2k-1}}$.

Solution: Rewriting the series as

$$\sum_{k=2}^{\infty} \frac{3^k}{2^{2k-1}} = \sum_{k=2}^{\infty} \frac{3^k}{2^{2k}\cdot 2^{-1}} = \sum_{k=2}^{\infty} \frac{2\cdot 3^k}{4^k} = \sum_{k=2}^{\infty} 2\Big(\frac{3}{4}\Big)^k,$$

we recognize that this is a geometric series with $r = 3/4$. Since $|r| < 1$, the series converges. The sum of a convergent geometric series is the first term of the series divided by the quantity $1 - r$, where r is the common ratio. The first term in our case corresponds to $k = 2$ so we have

$$\sum_{k=2}^{\infty} \frac{3^k}{2^{2k-1}} = \sum_{k=2}^{\infty} 2\Big(\frac{3}{4}\Big)^k = \frac{2(9/16)}{1-(3/4)} = \frac{9}{2}.$$

Example 4: Find the sum of the series $\sum_{k=0}^{\infty} \frac{4^{k-1}+7^{k+1}}{9^k}$.

Solution: Using part (b) of Theorem 3.10 and the formula for the sum of a geometric series, we find that

$$\sum_{k=0}^{\infty} \frac{4^{k-1}+7^{k+1}}{9^k} = \sum_{k=0}^{\infty} \frac{4^{k-1}}{9^k} + \sum_{k=0}^{\infty} \frac{7^{k+1}}{9^k} = \frac{1/4}{1-(4/9)} + \frac{7}{1-(7/9)} = \frac{9}{20} + \frac{63}{2} = 31.95.$$

As in this example, you should be able to determine the first term and the common ratio of a geometric series without doing a lot of extra work.

Example 5: Given that $\sum_{k=1}^{\infty} \frac{1}{k2^k} = \ln 2$, find the sum of the series $\sum_{k=3}^{\infty} \frac{3k+5}{k2^k}$.

Solution: We need to be careful here because the given series and the series in question start at different k values. We begin by noting that

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2} + \frac{1}{8} + \sum_{k=3}^{\infty} \frac{1}{k2^k} \quad \text{and thus} \quad \sum_{k=3}^{\infty} \frac{1}{k2^k} = \ln 2 - \frac{5}{8}.$$

(We have just pulled the first two terms out of the series.) Using part (b) of Theorem 3.10 and the formula for the sum of a geometric series, we obtain

$$\sum_{k=3}^{\infty} \frac{3k+5}{k2^k} = \sum_{k=3}^{\infty} \frac{3}{2^k} + \sum_{k=3}^{\infty} \frac{5}{k2^k} = \frac{3/8}{1-(1/2)} + 5\Big(\ln 2 - \frac{5}{8}\Big) = \frac{3}{4} + 5\ln 2 - \frac{25}{8} = 5\ln 2 - \frac{19}{8} \approx 1.090736.$$

Example 6: Find the sum of the series $18 - 12 + 8 - \frac{16}{3} + \frac{32}{9} - \frac{64}{27} + \cdots$.

Solution: Whenever a problem asks you to find the sum of a series, you should first check to see whether or not the series happens to be geometric. Since the ratios $-12/18$ and $8/(-12)$ of consecutive terms are both equal to $-2/3$, we see (after perhaps checking the next couple of terms) that the given series is a geometric series with common ratio $-2/3$. It follows that

$$18 - 12 + 8 - \frac{16}{3} + \frac{32}{9} - \frac{64}{27} + \cdots = \frac{18}{1-(-2/3)} = \frac{54}{5}.$$

(Do be careful when the common ratio r is negative.)

Example 7: Suppose that the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$ is given by $\{s_n\} = \Big\{\frac{5n}{2n+1}\Big\}$. Find a_{27} and the sum of the series.

Solution: By definition, the sum of a series is the limit of its sequence of partial sums. Thus

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5n}{2n+1} = \frac{5}{2}.$$

To find a_{27}, we simply compute (make certain you understand why this works)

$$a_{27} = s_{27} - s_{26} = \frac{5\cdot 27}{55} - \frac{5\cdot 26}{53} = \frac{27}{11} - \frac{130}{53} = \frac{1}{583}.$$

In general, we can use this idea to find a_n for any $n > 1$:

$$a_n = s_n - s_{n-1} = \frac{5n}{2n+1} - \frac{5n-5}{2n-1} = 5\Big(\frac{n(2n-1)-(n-1)(2n+1)}{(2n-1)(2n+1)}\Big) = \frac{5}{(2n-1)(2n+1)}.$$

Finally, note that $a_1 = s_1$ for any series of the form $\sum_{k=1}^{\infty} a_k$.

Extra Notes for Section 3.6

Example 1: Use the Integral Test to determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converges.

Solution: Consider the function f defined by $f(x) = \dfrac{1}{x^{5/2}}$. This function is continuous and decreasing on the interval $[1, \infty)$. Computing the improper integral of f on $[1, \infty)$ yields

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^{5/2}}\,dx &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^{5/2}}\,dx \\ &= \lim_{b\to\infty} -\frac{2}{3}\,x^{-3/2}\Big|_1^b \\ &= \lim_{b\to\infty} -\frac{2}{3}\Big(\frac{1}{b^{3/2}} - 1\Big) = \frac{2}{3}. \end{aligned}$$

Since the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^{5/2}}\,dx$ converges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converges by the Integral Test. We can use the equations next to the figure in Section 3.6 to estimate the value of the sum of this series. Referring to the last inequality that is adjacent to the figure and letting n go to infinity, we see that (noting that $a_1 = 1$ for our particular series)

$$\int_1^{\infty} \frac{1}{x^{5/2}}\,dx \le \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \le 1 + \int_1^{\infty} \frac{1}{x^{5/2}}\,dx \quad \text{and thus} \quad \frac{2}{3} \le \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \le \frac{5}{3}.$$

With the aid of technology (many options), we can find some partial sums of this series and get a better approximation to the sum. For example,

$$s_{1000} = \sum_{k=1}^{1000} \frac{1}{k^{5/2}} \approx 1.3414662.$$

The one potential difficulty with this calculation is that we do not know how accurate it is. Another calculation gives $s_{2000} \approx 1.3414798$. We can thus (somewhat cautiously) conclude that the sum of this series is about 1.3415.

Example 2: Use the Integral Test to determine whether or not the series $\displaystyle\sum_{k=3}^{\infty} \frac{4}{\sqrt[3]{2k-5}}$ converges.

Solution: Consider the function f defined by $f(x) = \dfrac{4}{\sqrt[3]{2x-5}}$. This function is continuous and decreasing on the interval $[3, \infty)$. (We can still apply the Integral Test but we need to use $[3, \infty)$ rather than $[1, \infty)$ to be certain that the values are positive.) Computing the appropriate improper integral in this case yields

$$\begin{aligned} \int_3^{\infty} \frac{4}{\sqrt[3]{2x-5}}\,dx &= \lim_{b\to\infty} \int_3^{b} \frac{4}{(2x-5)^{1/3}}\,dx \\ &= \lim_{b\to\infty} 3(2x-5)^{2/3}\Big|_3^b \\ &= \lim_{b\to\infty} \Big(3(2b-5)^{2/3} - 3\Big) = \infty. \end{aligned}$$

Since the improper integral $\displaystyle\int_3^{\infty} \frac{4}{\sqrt[3]{2x-5}}\,dx$ diverges, the series $\displaystyle\sum_{k=3}^{\infty} \frac{4}{\sqrt[3]{2k-5}}$ diverges by the Integral Test.

Example 3: Estimate a value of n for which $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 100$.

Solution: Since the given series is a p-series with $p \le 1$, we know that the series diverges. According to the inequalities listed by the figure in Section 3.6, we find that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \ge \int_1^n \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\Big|_1^n = 2\sqrt{n} - 2.$$

It is thus sufficient to find a value of n for which $2\sqrt{n} - 2 > 100$. It is easy to see that $n > (51)^2 = 2601$ will work. In other words, we know that

$$\sum_{k=1}^{2602} \frac{1}{\sqrt{k}} > 100.$$

Using some technology, we can actually compute these values more carefully and find that

$$\sum_{k=1}^{2602} \frac{1}{\sqrt{k}} \approx 100.5691 \quad \text{while} \quad \sum_{k=1}^{2573} \frac{1}{\sqrt{k}} \approx 99.9989971 \quad \text{and} \quad \sum_{k=1}^{2574} \frac{1}{\sqrt{k}} \approx 100.0187075.$$

The estimates provided by the Integral Test are important for both practical and theoretical purposes, but we will not spend much time on this aspect of series.

Example 4: For each positive integer n, let $y_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} - \displaystyle\int_1^n \frac{dx}{\sqrt{x}}$. Prove that $\{y_n\}$ is a decreasing sequence.

Solution: We want to prove that $y_{n+1} \le y_n$ for each positive integer n. This is equivalent to proving that $y_n - y_{n+1} \ge 0$. Writing out this expression and using properties of integrals, we find that

$$\begin{aligned}
y_n - y_{n+1} &= \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - \int_1^n \frac{dx}{\sqrt{x}}\right) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}} - \int_1^{n+1} \frac{dx}{\sqrt{x}}\right) \\
&= -\int_1^n \frac{dx}{\sqrt{x}} - \frac{1}{\sqrt{n+1}} + \int_1^{n+1} \frac{dx}{\sqrt{x}} \\
&= \int_n^{n+1} \frac{dx}{\sqrt{x}} - \frac{1}{\sqrt{n+1}}.
\end{aligned}$$

Since the function $1/\sqrt{x}$ is decreasing, we can use property (5) of integrals (see Section 2.5) to find that

$$y_n - y_{n+1} = \int_n^{n+1} \frac{dx}{\sqrt{x}} - \frac{1}{\sqrt{n+1}} \ge \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+1}} = 0.$$

(Alternatively, we can draw a picture and look at areas as in the text; the area under the curve is larger than the area of the inscribed rectangle.) We have thus shown that $y_n \ge y_{n+1}$ for each positive integer n. It follows that $\{y_n\}$ is a decreasing sequence. Note that, even though we proved that an equation was valid for all positive integers n, there was no need to use the Principle of Mathematical Induction. We chose the subtraction form for a decreasing sequence since the expression for y_n involved a sum. Finally, referring once again to the inequalities listed by the figure in Section 3.6, we see that each of the terms y_n is positive. By the Completeness Axiom, the sequence $\{y_n\}$ converges.

Extra Notes for Section 3.7

Example 1: Use the Comparison Test to determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^2-6}$ converges.

Solution: Looking at just the highest powers of k, this series looks a bit like $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, which is a divergent p-series. This leads us to believe that the series diverges so we need to show that its terms are larger than the terms of a divergent series of positive numbers. We can make fractions smaller by making their denominators larger. In this case, we note that

$$\frac{k}{7k^2-6} > \frac{k}{7k^2} = \frac{1}{7k} = \frac{1}{7} \cdot \frac{1}{k}$$

for all $k \geq 1$. Since the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{7k}$ diverges (it is a multiple of a divergent p-series), the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^2-6}$ diverges by the Comparison Test.

Example 2: Use the Comparison Test to determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^3-6}$ converges.

Solution: Looking at just the highest powers of k, this series looks a bit like $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p-series. This leads us to believe that the series converges so we need to show that its terms are smaller than the terms of a convergent series. We can make positive fractions larger by making their denominators smaller. We also want to do this in a way that makes it possible to combine like terms. In this case, we note that

$$\frac{k}{7k^3-6} < \frac{k}{7k^3-6k^3} = \frac{k}{k^3} = \frac{1}{k^2} \quad \text{or perhaps} \quad \frac{k}{7k^3-6} < \frac{k}{7k^3-k^3} = \frac{k}{6k^3} = \frac{1}{6k^2}.$$

The first inequality is valid for all $k \geq 1$, while the second is valid for all $k \geq 2$ (since we need k^3 to be larger than 6). In either case, since the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ (or one of its multiples) converges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^3-6}$ converges by the Comparison Test.

Example 3: Use the Comparison Test to determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{4^k+5^k}{3^k+6^k}$ converges.

Solution: Looking at the exponential terms with the larger bases, this series looks a bit like $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^k$, which is a convergent geometric series. Noting that

$$\frac{4^k+5^k}{3^k+6^k} < \frac{5^k+5^k}{3^k+6^k} < \frac{5^k+5^k}{6^k} = 2 \cdot \frac{5^k}{6^k} = 2\left(\frac{5}{6}\right)^k$$

for all $k \geq 1$ and that the series $\displaystyle\sum_{k=1}^{\infty} 2\left(\frac{5}{6}\right)^k$ is a convergent geometric series (the absolute value of the common ratio is less than 1), the series $\displaystyle\sum_{k=1}^{\infty} \frac{4^k+5^k}{3^k+6^k}$ converges by the Comparison Test.

Example 4: Use the Limit Comparison Test to determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k+3}{k^2+4k-2}$ converges.

Solution: For large values of k, the terms of this series resemble those of the series $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k}$. This series diverges (since it is a multiple of a divergent p-series) and we find that

$$\alpha = \lim_{k\to\infty} \frac{\dfrac{2k+3}{k^2+4k-2}}{\dfrac{2}{k}} = \lim_{k\to\infty} \frac{2k^2+3k}{2k^2+8k-4} = 1.$$

Since $\alpha > 0$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k+3}{k^2+4k-2}$ diverges by the Limit Comparison Test. (We can certainly use $1/k$ instead of $2/k$ and reach the same conclusion; obtaining just a slightly different positive value for α.)

Example 5: Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{7k^2+3k-4}{2k^4+4k^3-2}$ converges.

Solution: For large values of k, the terms of this series resemble those of a multiple of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$. We thus predict that this series converges. (It is important for you to be able to make such conjectures quickly AND it is necessary that you know how to back up your conjectures with a proof if necessary.) To prove this fact, we can use either the Comparison Test or the Limit Comparison Test. To use the LCT, we compute the limit (skipping straight to the invert and multiply step):

$$\alpha = \lim_{k\to\infty} \left(\frac{7k^2+3k-4}{2k^4+4k^3-2} \cdot \frac{k^2}{1} \right) = \lim_{k\to\infty} \frac{7k^4+3k^3-4k^2}{2k^4+4k^3-2} = \frac{7}{2}.$$

Since $\alpha < \infty$ and the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{7k^2+3k-4}{2k^4+4k^3-2}$ converges by the Limit Comparison Test. On the other hand, we can be very careful with inequalities and note that

$$\frac{7k^2+3k-4}{2k^4+4k^3-2} < \frac{7k^2+k^2}{2k^4+4k^3-2} < \frac{7k^2+k^2}{2k^4} = \frac{8k^2}{2k^4} = \frac{4}{k^2}$$

for all $k \geq 1$. Since $\displaystyle\sum_{k=1}^{\infty} \frac{4}{k^2}$ converges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{7k^2+3k-4}{2k^4+4k^3-2}$ converges by the Comparison Test.

Example 6: Estimate (without technology) the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k+4^k}$.

Solution: Using some simple over and under estimates, we see that

$$\frac{2^k}{4^k+4^k} < \frac{2^k}{3^k+4^k} < \frac{2^k}{3^k+3^k} \quad \text{or} \quad \frac{1}{2}\left(\frac{1}{2}\right)^k < \frac{2^k}{3^k+4^k} < \frac{1}{2}\left(\frac{2}{3}\right)^k$$

for all $k \geq 1$. Using the formula for the sum of a geometric series, we find that

$$\frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^k < \sum_{k=1}^{\infty} \frac{2^k}{3^k+4^k} < \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^k = 1.$$

Hence, the sum of the series is between 0.5 and 1. (Technology gives the actual sum as about 0.633754.)

Extra Notes for Section 3.8

Recall that the main goal of the last few sections is to determine whether or not the series $\sum_{k=1}^{\infty} a_k$ converges. That is, we want to know if there is a number represented by the infinite sum

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots.$$

We have seen that this occurs in some cases and not in others. The tests that we have developed thus far (with paraphrased informal synopses) to help us decide are the Divergence Test (if the terms a_k do not have a limit of 0, then the series diverges), the Integral Test (the series and the corresponding improper integral behave in the same way), the Comparison Test (less than convergent is convergent and greater than divergent is divergent), and the Limit Comparison Test (if the limit of the ratios of the terms exists, then the series do the same thing). Using these ideas, we have learned that

i. geometric series $\displaystyle\sum_{k=0}^{\infty} ar^k$ converge when $|r| < 1$ and diverge when $|r| \geq 1$;

ii. p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge when $p > 1$ and diverge when $0 < p \leq 1$.

Furthermore, for geometric series, we know that the sum of the series is $a/(1-r)$, that is, the sum is the first term divided by 1 minus the common ratio. For p-series, there are no simple formulas for the sum when the series converges (although some of the sums are known); we simply know that the series converges and we can then use technology to find an approximation for the sum.

The latter three tests require series with positive terms and this can be a serious drawback for certain series. The concept of absolute convergence is helpful in some cases: if $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ must also converge. Since the terms of $\sum_{k=1}^{\infty} |a_k|$ are positive (or at least nonnegative), we can use one of the comparison tests. However, the next three tests that we consider (Alternating Series Test, Ratio Test, and Root Test) allow us to look at series with infinitely many positive and negative terms.

Example 1: Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(3k)}{k^{4/3}}$ converges.

Solution: Using basic properties of the sine function, we see that

$$\left|\frac{\sin(3k)}{k^{4/3}}\right| = \frac{|\sin(3k)|}{k^{4/3}} \leq \frac{1}{k^{4/3}}$$

for all positive integers k. Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$ is a convergent p-series, the series $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\sin(3k)}{k^{4/3}}\right|$ converges by the Comparison Test. It follows that the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(3k)}{k^{4/3}}$ converges.

Example 2: Determine whether or not the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{2k-1}$ converge.

Solution: These two series behave in the same way since one is just a constant multiple of the other. It is easy to see that the sequence $\{1/(2k-1)\}$ is decreasing (the denominator increases with k while the numerator stays the same) and converges to 0. We conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ converges by the Alternating Series Test. It then follows that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{2k-1} = (-1)\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ converges as well.

This is perhaps a good time to recall various ways to prove that a sequence is monotone. Suppose we want to prove that a sequence $\{a_k\}$ is decreasing. There are several ways to do so.

a. Use algebra to prove that $a_k \geq a_{k+1}$ for all positive integers k.
b. Use algebra to prove that $a_k - a_{k+1} \geq 0$ for all positive integers k.
c. Use algebra to prove that $a_{k+1}/a_k \leq 1$ for all positive integers k, ASSUMING all terms are positive.
d. Use induction to prove that $a_n \geq a_{n+1}$ for all positive integers n.
e. Find a function f such that $f(k) = a_k$ for each k and show that $f'(x) \leq 0$ for all $x \geq 1$.

Methods for proving that a sequence is increasing are analogous (see the extra notes for Section 3.4).

Example 3: Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{7k^2-6}$ is conditionally convergent.

Solution: We first consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^2-6}$, which consists of the absolute value of each term. Referring to Example 1 in the extra notes for Section 3.7, we see that this series diverges. (If you are confronted with a new series, you would need to include the details here; these solutions do get a bit lengthy.) To verify that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{7k^2-6}$ converges, we use the Alternating Series Test. Ignoring the $(-1)^{k+1}$ term, we consider the sequence $\left\{\dfrac{k}{7k^2-6}\right\}$. This sequence clearly converges to 0 and has positive terms. To prove that it is decreasing, let f be the function defined by

$$f(x) = \frac{x}{7x^2-6} \quad \text{and note that} \quad f'(x) = \frac{7x^2-6-x(14x)}{(7x^2-6)^2} = \frac{-7x^2-6}{(7x^2-6)^2}.$$

Since $f'(x) < 0$ for all $x \geq 1$, the function f is decreasing on $[1,\infty)$. It follows that $\left\{\dfrac{k}{7k^2-6}\right\}$ is a decreasing sequence. By the Alternating Series Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{7k^2-6}$ converges. Finally, since the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{7k^2-6}$ diverges and the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{7k^2-6}$ converges, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{7k^2-6}$ is a conditionally convergent series.

Extra Notes for Section 3.9

Example 1: Determine whether or not the series $\sum_{k=1}^{\infty}\Big(\frac{2k+17}{3k-100}\Big)^k$ converges.

Solution: Since the form of the terms involves k as part of the exponent, the Root Test seems like a good choice. Note that

$$\ell = \lim_{k\to\infty}\sqrt[k]{\Big|\Big(\frac{2k+17}{3k-100}\Big)^k\Big|} = \lim_{k\to\infty}\Big|\frac{2k+17}{3k-100}\Big| = \frac{2}{3}.$$

Since $\ell < 1$, the series $\sum_{k=1}^{\infty}\Big(\frac{2k+17}{3k-100}\Big)^k$ converges absolutely by the Root Test.

Example 2: Determine whether or not the series $\sum_{k=1}^{\infty}\Big(\frac{2k-1}{2k}\Big)^{k^2}$ converges.

Solution: As with Example 1, the Root Test seems like a good choice. In this case, we find that

$$\ell = \lim_{k\to\infty}\sqrt[k]{\Big|\Big(\frac{2k-1}{2k}\Big)^{k^2}\Big|} = \lim_{k\to\infty}\Big(\frac{2k-1}{2k}\Big)^k = \lim_{k\to\infty}\Big(1+\frac{-1/2}{k}\Big)^k = e^{-1/2}$$

(see part 7 of Theorem 3.8). Since $\ell < 1$, the series $\sum_{k=1}^{\infty}\Big(\frac{2k-1}{2k}\Big)^{k^2}$ converges absolutely by the Root Test.

Example 3: Determine whether or not the series $\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{3^k(k!)^2}$ converges.

Solution: Since the terms of this series involve factorials, it is best to use the Ratio Test. The kth term and $(k+1)$st term of the sequence of terms are

$$\frac{(-1)^k(2k)!}{3^k(k!)^2} \quad\text{and}\quad \frac{(-1)^{k+1}(2k+2)!}{3^{k+1}((k+1)!)^2},$$

respectively. Be certain that you notice what happens when $k+1$ is substituted for k and be very careful with parentheses as, for instance, the number $2k!$ is very different from the number $(2k)!$. Omitting the invert and multiply step, we find that

$$\ell = \lim_{k\to\infty}\Big|\frac{(-1)^{k+1}(2k+2)!}{3^{k+1}((k+1)!)^2}\cdot\frac{3^k(k!)^2}{(-1)^k(2k)!}\Big| = \lim_{k\to\infty}\frac{(2k+2)(2k+1)}{3(k+1)^2} = \frac{4}{3}.$$

Since $\ell > 1$, the series $\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{3^k(k!)^2}$ diverges by the Ratio Test.

It is very important that you understand how the factorial terms cancel. For instance,

$$\frac{(2k+2)!}{(2k)!} = \frac{(2k+2)(2k+1)(2k)(2k-1)\cdots 3\cdot 2\cdot 1}{(2k)(2k-1)\cdots 3\cdot 2\cdot 1} = (2k+2)(2k+1)$$

or preferably (indicating a better understanding)

$$\frac{(2k+2)!}{(2k)!} = \frac{(2k+2)(2k+1)(2k)!}{(2k)!} = (2k+2)(2k+1).$$

Example 4: Determine whether or not the series $\sum_{k=1}^{\infty} \frac{(-1)^k(k^2+3k)}{k^4-5k+7}$ converges absolutely, converges conditionally, or diverges.

Solution: Since the numerator and denominator of the fraction that makes up the terms of this series involve polynomials, the Ratio Test and Root Test will give no information. To see why this happens, note that

$$\ell = \lim_{k\to\infty}\left(\frac{(k+1)^2+3(k+1)}{(k+1)^4-5(k+1)+7}\cdot\frac{k^4-5k+7}{k^2+3k}\right) = 1;$$

$$\ell = \lim_{k\to\infty}\frac{\sqrt[k]{k^2+3k}}{\sqrt[k]{k^4-5k+7}} = 1.$$

The first limit is 1 due to the fact that the polynomials in the fraction are both of degree 6 with equal leading coefficients. The second limit is 1 as a consequence of part 6 of Theorem 3.8. It is very important that you recognize this in advance and that you do not then waste your time doing these tests when you know they will give you no information. We can, however, apply the Limit Comparison Test to the series $\sum_{k=1}^{\infty}\frac{k^2+3k}{k^4-5k+7}$. The terms of this series are similar to $1/k^2$ and we find that

$$\lim_{k\to\infty}\left(\frac{k^2+3k}{k^4-5k+7}\cdot\frac{k^2}{1}\right) = 1.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series, the series $\sum_{k=1}^{\infty}\left|\frac{(-1)^k(k^2+3k)}{k^4-5k+7}\right|$ converges by the Limit Comparison Test. It follows that the series $\sum_{k=1}^{\infty}\frac{(-1)^k(k^2+3k)}{k^4-5k+7}$ is absolutely convergent.

Example 5: Find all the values of x for which the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^3}{3^k}(x+1)^k$ converges.

Solution: We apply the Ratio Test and find that

$$\ell = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(k+1)^3(x+1)^{k+1}}{3^{k+1}}\cdot\frac{3^k}{(-1)^{k+1}k^3(x+1)^k}\right| = \lim_{k\to\infty}\left(\frac{(k+1)^3}{3k^3}\,|x+1|\right) = \frac{|x+1|}{3}.$$

We know that $\ell < 1$ is needed to guarantee convergence so we want values of x that satisfy $|x+1| < 3$, which is equivalent to $-3 < x+1 < 3$ or $-4 < x < 2$. Hence, the series converges for all x in the interval $(-4, 2)$. When considering series like this one, be very careful to include the absolute value of the terms when performing either the Ratio Test or the Root Test.

Extra Notes for Section 3.10

Example 1: Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{4k+2}{3^k}\,(x-1)^k$.

Solution: We apply the Ratio Test and find that

$$\ell = \lim_{k\to\infty}\left|\frac{(4k+6)(x-1)^{k+1}}{3^{k+1}} \cdot \frac{3^k}{(4k+2)(x-1)^k}\right| = \lim_{k\to\infty}\left(\frac{4k+6}{3(4k+2)}\,|x-1|\right) = \frac{|x-1|}{3}.$$

We know the power series converges for those values of x that make $\ell < 1$. In this case, we find that $|x-1| < 3$. This means that the series converges for all values of x that are within three units of the center 1. It follows that the radius of convergence of the power series is 3.

Example 2: Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 5^k}{(2k)!}\,x^k$.

Solution: We apply the Ratio Test and find that

$$\ell = \lim_{k\to\infty}\left|\frac{5^{k+1}x^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{5^k x^k}\right| = \lim_{k\to\infty} \frac{5|x|}{(2k+2)(2k+1)} = 0$$

for any value of x. It follows that the radius of convergence of the power series is ∞.

Example 3: Find the interval of convergence for the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(3k+1)}\,(x-1)^k$.

Solution: We apply the Ratio Test and find that

$$\ell = \lim_{k\to\infty}\left|\frac{(x-1)^{k+1}}{2^{k+1}(3k+4)} \cdot \frac{2^k(3k+1)}{(x-1)^k}\right| = \lim_{k\to\infty}\left(\frac{3k+1}{2(3k+4)}\,|x-1|\right) = \frac{|x-1|}{2}.$$

It follows that the power series converges for those values of x that satisfy $|x-1| < 2$ or $-1 < x < 3$. However, the Ratio Test does not give us any information about the behavior of the power series at the endpoints (where the value of ℓ is 1). Checking these values separately, we find that

$$x = -1 \quad \text{gives} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(3k+1)}\,(-2)^k = \sum_{k=0}^{\infty} \frac{1}{3k+1}, \quad \text{which diverges by the Comparison Test;}$$

$$x = 3 \quad \text{gives} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(3k+1)}\,(2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+1}, \quad \text{which converges by the Alternating Series Test.}$$

The interval of convergence for this power series is thus $(-1, 3]$. As mentioned in the textbook discussion of power series, this means that the domain of the function f defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(3k+1)}\,(x-1)^k$$

is the set of all x values that satisfy $-1 < x \leq 3$.

Example 4: Determine in more familiar terms the function represented by $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{7^k} (x-2)^k$.

Solution: Since we are asked to find the sum of a series, we need to think about which series we have previously learned that allow us to find the sum. A little thought (or page turning through the textbook) reveals that the only series we have seen thus far that come with a formula for the sum are the geometric series. So, we hope that this series is actually a geometric series. Given the form of the terms (all the k values appear as exponents), we see that the series is indeed geometric. The formula for the sum is the first term divided by 1 minus the common ratio. It follows that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{7^k} (x-2)^k = \frac{\frac{x-2}{7}}{1 - \left(-\frac{x-2}{7}\right)} = \frac{x-2}{x+5}.$$

Recall that a geometric series converges when $|r| < 1$. In our case, we have used $-(x-2)/7$ as the value for r. It follows that the sum of the series is only valid for those values of x that satisfy

$$\left|-\frac{x-2}{7}\right| < 1 \quad \Leftrightarrow \quad |x-2| < 7.$$

This fact indicates that the radius of convergence of the power series must be 7. It also shows that the domain of the function f defined by the power series is $-5 < x < 9$. As a final point, notice that the simplified formula for $f(x)$ is not defined at $x = -5$ and the distance from this point to the center $x = 2$ of the power series is 7, the same as the radius of convergence of the power series. As we will see, this is no accident.

Extra Notes for Section 3.11

Example 1: Use a geometric series to find a power series expression centered at 0 for $f(x) = \dfrac{4}{3x+1}$.

Solution: Using the familiar formula for the sum of a geometric series (first term divided by 1 minus the common ratio as long as the absolute value of the common ratio is less than 1), we find that

$$\frac{4}{3x+1} = \frac{4}{1-(-3x)} = \sum_{k=0}^{\infty} 4(-3x)^k$$

assuming that $|-3x| < 1$. It follows that a power series expression centered at 0 for the function f is

$$f(x) = \sum_{k=0}^{\infty} 4(-1)^k 3^k x^k,$$

which is valid for all x that satisfy $|x| < 1/3$.

Example 2: Use a geometric series to find a power series expression centered at 0 for $f(x) = \dfrac{x}{3x+2}$.

Solution: Proceeding as in the previous example (but with the recognition that a 1 is needed in the denominator), we find that

$$\frac{x}{3x+2} = \frac{x/2}{1-(-3x/2)} = \sum_{k=0}^{\infty} \frac{x}{2}\left(-\frac{3x}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{2^{k+1}}\, x^{k+1}$$

assuming that $|x| < 2/3$. Since the final form of the above power series involves the $k+1$ exponent, we can shift the indices to obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{2^{k+1}}\, x^{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 3^{k-1}}{2^k}\, x^k.$$

The exponents are reduced by 1 to compensate for the sum beginning at a higher number. If you do not believe that the two series are the same, write out the first three terms of each of them and see that they are indeed the same sum. It follows that a power series expression centered at 0 for the function f is

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 3^{k-1}}{2^k}\, x^k.$$

which is valid for all x that satisfy $|x| < 2/3$. (Note that $(-1)^{k-1} = (-1)^{k+1}$ for all values of k.)

Example 3: Use a geometric series to find a power series expression centered at 0 for $f(x) = \dfrac{2}{(3x+1)^2}$.

Solution: Because of the square in the denominator, we must do some work to see how a geometric series comes into play. As with one of the examples in the textbook, we take the derivative of the appropriate function to find that

$$f(x) = \frac{2}{(3x+1)^2} = -\frac{2}{3} \cdot \frac{d}{dx}\left(\frac{1}{3x+1}\right).$$

Using the geometric series, we find that

$$\frac{1}{3x+1} = \frac{1}{1-(-3x)} = \sum_{k=0}^{\infty}(-3x)^k = \sum_{k=0}^{\infty}(-1)^k 3^k x^k$$

and thus

$$f(x) = -\frac{2}{3}\cdot\frac{d}{dx}\sum_{k=0}^{\infty}(-1)^k 3^k x^k = -\frac{2}{3}\sum_{k=1}^{\infty}(-1)^k 3^k k x^{k-1} = \sum_{k=1}^{\infty} 2(-1)^{k+1}3^{k-1}kx^{k-1}.$$

(The starting index changed from $k=0$ to $k=1$ because the derivative of the first term is 0.) As in Example 2, we can shift the index to obtain

$$f(x) = \sum_{k=0}^{\infty} 2(-1)^k 3^k (k+1)x^k.$$

This power series expression for the function f is valid for all x that satisfy $|x| < 1/3$.

Example 4: Use a geometric series to find a power series expression centered at 2 for $f(x) = \dfrac{1}{3x+1}$.

Solution: Performing some creative algebra, we can make a multiple of $x-2$ appear as the r value for the geometric series. Doing so yields

$$f(x) = \frac{1}{3x+1} = \frac{1}{3(x-2)+7} = \frac{\frac{1}{7}}{1-\left(\frac{-3(x-2)}{7}\right)} = \sum_{k=0}^{\infty}\frac{1}{7}\left(\frac{-3(x-2)}{7}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k 3^k}{7^{k+1}}(x-2)^k.$$

It is important that you learn how to read these sorts of equations. Look at each equals sign and convince yourself how each step was carried out. Note that the series converges for all values of x that satisfy $|x-2| < 7/3$ and that the distance from the center 2 to the undefined point $-1/3$ of the function is also $7/3$.

Example 5: Find the sum of the series $\displaystyle\sum_{k=1}^{\infty}\frac{5k-3}{4^k}$.

Solution: We begin by using the linearity properties of series and some simplification to obtain

$$\sum_{k=1}^{\infty}\frac{5k-3}{4^k} = 5\sum_{k=1}^{\infty}\frac{k}{4^k} - 3\sum_{k=1}^{\infty}\frac{1}{4^k} = 5\sum_{k=1}^{\infty}k\left(\frac{1}{4}\right)^k - 3\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k.$$

The second series is a geometric series so we know how to find its sum. The first series is of the form

$$\sum_{k=1}^{\infty}kx^k = \frac{x}{(1-x)^2},$$

whose formula was derived in this section. It follows that

$$\sum_{k=1}^{\infty}\frac{5k-3}{4^k} = 5\sum_{k=1}^{\infty}k\left(\frac{1}{4}\right)^k - 3\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k = 5\left(\frac{1/4}{(3/4)^2}\right) - 3\left(\frac{1/4}{3/4}\right) = \frac{20}{9} - 1 = \frac{11}{9}.$$

Extra Notes for Section 3.12

Example 1: Use known Maclaurin series to determine in more familiar terms the functions represented by the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}k!}\, x^k$ and $\displaystyle\sum_{k=0}^{\infty} \frac{(-6)^k}{(2k+1)!}\, x^{2k}$.

Solution: Since the first series involves $k!$, we conjecture that the relevant function is related to e^x. Writing the series in the appropriate form, we find that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}k!}\, x^k = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1/2)^k}{k!}\, x^k = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x}{2}\right)^k = \frac{1}{2}\, e^{-x/2}.$$

The second series has the factorials of odd integers and this leads us to look for an appropriate sine function. Performing some algebra yields

$$\sum_{k=0}^{\infty} \frac{(-6)^k}{(2k+1)!}\, x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\sqrt{6}\, x\right)^{2k} = \frac{1}{\sqrt{6}\, x}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\sqrt{6}\, x\right)^{2k+1} = \frac{1}{\sqrt{6}\, x}\, \sin\!\left(\sqrt{6}\, x\right).$$

Pay attention to the steps required to express the sum in the appropriate form. Ask questions, either in class or during office hours, for any steps that you do not understand.

Example 2: Find $f^{(101)}(0)$ for the function f defined by $f(x) = xe^{-x^2/2}$.

Solution: We start by finding the Maclaurin series for f. Using the known Maclaurin series for e^x, we find that

$$f(x) = xe^{-x^2/2} = x\sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x^2}{2}\right)^k = x\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}\, x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}\, x^{2k+1}.$$

We know from the theory of power series (for the case when the center is 0) that $f^{(k)}(0) = k!\, c_k$, where c_k represents the coefficient on the term x^k. For our situation, the term x^{101} appears when $k = 50$. It follows that the coefficient on this power of x, which we denote by c_{101}, is given by

$$c_{101} = \frac{(-1)^{50}}{2^{50}50!} = \frac{1}{2^{50}50!}.$$

Therefore, we find that

$$f^{(101)}(0) = 101!\, c_{101} = \frac{101!}{2^{50}50!}.$$

Pay particular attention to how the k values were used in this calculation. Since these numbers are so large (the final integer involves 81 digits), we do not simplify them any further.

Example 3: Find the Taylor series centered at 4 for the function $f(x) = 1/\sqrt{x}$.

Solution: Since the given function is not one that has been considered thus far, we have to return to the definition and actually find the coefficients. The first step is to write out some derivatives and try to spot a pattern.

$$\begin{aligned}
f(x) &= x^{-1/2} \\
f'(x) &= -\frac{1}{2}\,x^{-3/2} \\
f''(x) &= \frac{3}{2}\cdot\frac{1}{2}\,x^{-5/2} \\
f^{(3)}(x) &= -\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,x^{-7/2} \\
f^{(4)}(x) &= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,x^{-9/2} \\
f^{(5)}(x) &= -\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,x^{-11/2}
\end{aligned}$$

for $k \geq 1$,

$$\begin{aligned}
f^{(k)}(x) &= (-1)^k \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^k}\,x^{-(2k+1)/2} \\
f^{(k)}(4) &= (-1)^k \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^k}\,4^{-(2k+1)/2} \\
f^{(k)}(4) &= (-1)^k \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^k\,2^{2k+1}}
\end{aligned}$$

As with the example in the textbook, when searching for a pattern, it is best not to multiply out terms. We also used the fact that $4^{1/2} = 2$. Now recall that the coefficients of the power series are given by

$$c_k = \frac{f^{(k)}(4)}{k!} = (-1)^k \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^{3k+1}k!}$$

for $k \geq 1$. When $k = 0$, we find that $c_0 = f(4)/0! = 1/2$. Since the first term of the Taylor series does not fit the pattern, it is pulled out separately. We thus obtain

$$\frac{1}{\sqrt{x}} = \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^k \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^{3k+1}k!}\,(x-4)^k.$$

Remember to write the series using powers of $x - a$, where a is the center of the power series. In this case, we need powers of $x - 4$.

As another observation, we can multiply by a messy version of 1 to obtain

$$\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{k!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot\cdots\cdot(2k-1)\cdot(2k)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)\,k!} = \frac{(2k)!}{2^k k!\cdot k!} = \frac{(2k)!}{2^k (k!)^2},$$

from which it follows that

$$c_k = \frac{(-1)^k(2k)!}{2^{4k+1}(k!)^2}.$$

Note that this formula conveniently gives $c_0 = 1/2$ so we no longer need to pull out the first term. We thus find that

$$\frac{1}{\sqrt{x}} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{2^{4k+1}(k!)^2}\,(x-4)^k.$$

Formulas such as this one may help explain some of the power series exercises in Section 3.10 that involved factorials or other similar products.