Math 126: Prelude to Chapter 3

Our next topic is sequences and series. Many students find this material to be the most difficult and challenging aspect of the entire calculus sequence (bad pun). This comment is not intended to scare you; rather it is intended to let you know that you will need to be working harder over the next few weeks. For a hiking analogy, you need to have a different mindset if you are taking a twenty mile hike up a peak with a 5000 foot elevation gain as opposed to a four mile walk around a lake. The following comments may help you prepare for the challenging content that lies ahead.

Let me start by saying (or shouting) that you MUST change your approach to mathematics as we work through sequences and series. The old and familiar method of learning a technique or two and then practicing it some will not work for this material. You must think deeply about the ideas and focus on the concepts. At the level of computation, you will see little connection between the problems that are assigned but if you look at the bigger picture and focus on the ideas, you will find the pieces fitting together. This approach will take a great deal of mental effort on your part and I cannot do it for you. I can offer some explanation and encouragement but you must do the work to incorporate these concepts into your mind. As a result, you will need to spend time (maybe a lot of time) thinking about mathematics, not just writing down numbers and the like. You will need to develop patience (both with yourself and possibly with the tutors) and a high tolerance for frustration. Hopefully, you will acquire a new and broader view of mathematics and learn how to think more abstractly.

Returning to the hiking analogy, I can tell you that there is a fantastic view from the top of the peak we are climbing (and there is) and encourage you to keep going even when the trail is steep and rocky. Several things may occur. You might get to the top and agree that the view is breathtaking and well worth the effort, you might get to the top and have the fog roll in and obstruct the view, or you may see the view and not be impressed in the least. In the latter cases, you can focus on the companionship (misery loves company and creates bonding), enjoy a few sights along the way (Fibonacci numbers are kind of cool), or just get a great aerobic workout. In any event, we have some hard work ahead of us and you need to be aware that you will need a different approach if you desire to succeed.

Our first topic will be proof by induction. This proof technique is often covered briefly in a precalculus course and some of the foundations of logic are mentioned in high school geometry courses. We present here a very brief review. In the conditional sentence $P \Rightarrow Q$ (if P, then Q), the sentence P is usually referred to as the hypothesis and the sentence Q is called the conclusion. By rearranging and/or negating P and Q, we can form various other conditionals related to $P \Rightarrow Q$. Beginning with the conditional $P \Rightarrow Q$, the converse is $Q \Rightarrow P$ (if Q, then P), the contrapositive is $\neg Q \Rightarrow \neg P$ (if not Q, then not P), and the inverse is $\neg P \Rightarrow \neg Q$ (if not P, then not Q). As an illustration, consider the following important theorem from differential calculus.

conditional:	If f is differentiable at c, then f is continuous at c.
converse:	If f is continuous at c, then f is differentiable at c.
contrapositive:	If f is not continuous at c, then f is not differentiable at c.
inverse:	If f is not differentiable at c, then f is not continuous at c.

It is a simple logical fact that a conditional and its contrapositive always have the same truth value. It is very important to note that the converse may or may not have the same truth value as the given conditional; the previous illustration provides one example where the truth values of a statement and its converse are not the same. (Be certain that you can give an example to show that the converse in this case is false.) It is a common mistake for students to turn theorems around without thinking much about it. Be aware of this potential pitfall and think carefully before drawing conclusions. The inverse, which is the contrapositive of the converse, is not referred to very often in mathematics.

If you pause to think about it, you will realize that every field of study (even discussions with your friends or parents) requires proof or some sort of criteria to establish valid arguments. For instance, consider the following five statements.

(1) Climate change is a consequence of human activities.

(2) Smoking increases the risk of contracting lung cancer.

(3) Stricter gun laws are needed to reduce the rate of violent crime.

(4) 12475 can be represented as a sum of two perfect squares.

(5) There are an infinite number of primes.

If you believe that a given statement is true or false, how can you convince someone that your point of view is correct? What sort of argument is sufficient? Is there such a thing as truth or Truth? Such questions lead into the realms of philosophy and psychology. We won't dig quite that deeply and simply accept the usual and well-known rules of logic as obvious truths. Mathematicians begin with these rules of logic, some undefined and defined terms, and a few axioms, then they build up a body of knowledge; knowledge that they are confident is true and that others operating with the same rules must also believe is true. We are just going to look at one simple type of proof; proof by induction. With it, we can prove an infinite number of statements (one for each positive integer n) in only two steps. You can read about this proof technique in Section 3.1. We will be spending two class periods talking about this proof technique and learning how to use it. For some further examples of induction proofs, you can go to http://people.whitman.edu/ gordon/ then click on the links Math 260 and Model Induction Proofs.

Finally, let me state Leonardo of Pisa's famous problem from his 1202 text *Liber Abbaci*:

> How many pairs of rabbits can be bred in one year from one pair? A certain person places one pair of rabbits in a certain place surrounded on all sides by a wall. We want to know how many pairs can be bred from that pair in one year, assuming it is their nature that each month they give birth to another pair, and in the second month, each new pair can also breed.

We will discuss this problem on our second day in Section 3.1.

Here are some further comments on mathematical induction that may be helpful. Consider the statement

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6},$$

where n is a positive integer. This type of $\cdots$ notation will appear frequently in Chapter 3. The idea is that the first few terms establish a pattern and the last term tells you when to stop. For this statement, when $n = 5$ we continue the (hopefully obvious) pattern until we get to the term $5 \cdot 7$ and obtain

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + 5 \cdot 7 = \frac{5 \cdot 6 \cdot 17}{6},$$

and when $n = 6$ we obtain

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 = \frac{6 \cdot 7 \cdot 19}{6}.$$

It is easy to verify that each of the first expressions equals 85 and each of the second expressions equals 133. The original equation also tells us that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$$

or

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + p(p+2) = \frac{p(p+1)(2p+7)}{6}$$

since the name of the integer (n, k, or p) does not matter. Furthermore, we can write

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (n+1)(n+3) = \frac{(n+1)(n+2)\big(2(n+1)+7\big)}{6}$$

(replacing n with $n+1$), or

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (2n-1)(2n+1) = \frac{(2n-1)(2n)\big(2(2n-1)+7\big)}{6}$$

(replacing n with $2n-1$), or even

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + p^2(p^2+2) = \frac{p^2(p^2+1)(2p^2+7)}{6}$$

(replacing n with p^2). This is possible since the variable n just holds a place. We should also point out what the original equation means when $n = 1$, $n = 2$, and $n = 3$; these yield

$$1 \cdot 3 = \frac{1 \cdot 2 \cdot 9}{6}, \quad 1 \cdot 3 + 2 \cdot 4 = \frac{2 \cdot 3 \cdot 11}{6}, \quad \text{and} \quad 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = \frac{3 \cdot 4 \cdot 13}{6},$$

respectively. In each case, the $n(n+2)$ term tells us when to stop the pattern and, in these cases, we do not have to go very far to reach the stop values. Using summation notation, we would have

$$\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6},$$

which may be easier to understand for some people.

When we claim that the original equation is true for all values of n, we are making an infinite number of assertions, one assertion for each positive integer n. If we let P_n be the statement

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6},$$

then our previous work shows that P_1, P_2, P_3, P_5, and P_6 are all true. It is easy to verify that P_4 is also true. However, this does not guarantee that P_n is true for every positive integer n; there are still an infinite number of integers that we have not checked. How do we know that P_{100} is true? We could ask a computer to help us verify that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + 100 \cdot 102 = \frac{100 \cdot 101 \cdot 207}{6}$$

and conclude that P_{100} is true, but very few computers would be able to verify that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + 10^{10000} \cdot \left(10^{10000} + 2\right) = \frac{10^{10000} \cdot \left(10^{10000} + 1\right) \cdot \left(2 \cdot 10^{10000} + 7\right)}{6},$$

and no computer can check all of the possible integer values for n. It is also not safe to assume that a pattern continues; there are many examples of statements that are true for the first million or more integers, then become false for larger integers.

So how can we know for certain that P_n is true for every positive integer n? This is where the Principle of Mathematical Induction comes into play. If we can prove that P_1 is true (which is very easy to do in most cases) and then prove the conditional "if P_k is true, then P_{k+1} is true", then we are guaranteed that the statement P_n is true for every positive integer n. In other words, we can state the PMI as follows.

Principle of Mathematical Induction: For each positive integer n, let P_n be a statement that depends on n. If P_1 is true and the conditional statement "if P_k, then P_{k+1}" is valid, then the statement P_n is true for each positive integer n.

The key step in an induction proof is proving the "if P_k, then P_{k+1}" statement. We make the assumption that P_k is true for some generic positive integer k, then try to use this fact to prove that P_{k+1} is true. It is this part of the proof that may be challenging since the connection between P_k and P_{k+1} may not be immediately clear. This part can sometimes be confusing for students since we are working with letters rather then numbers. For the problem under consideration, we assume that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$$

is true, then try to use this fact to prove that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}$$

is true. Since the details appear elsewhere (see the Model Induction Proofs in the Math 260 material), we do not include them here.