Math 244: Review for Final Exam

Our final exam is scheduled for Friday afternoon, December 16. The exam begins at 2 pm in our usual classroom, Olin 301. The exam is written for a two hour period, but you may have three hours for the exam should you choose to stay that long (and many students have done so in the past). This means that it will most likely be 5:00 pm and very dark when you finish.

The final exam is comprehensive, that is, it covers all of the material that has been discussed this semester. You might find it helpful to read the list of assignments for the semester and to flip through the sections of the textbook that we have covered. You need to be familiar with the basic concepts and techniques we have considered. This includes the standard formulas that frequently occur in the problems that we have been doing (such as the general form of spring-mass problems, basic Maclaurin series, formulas for Laplace transforms, etc.). You can also look over the four exams and the extra problems that have been assigned. If you want to look at some specific problems from the textbook, the following list can serve as a guide. It is probably not possible to do all of these problems again—there is simply not enough time to do so. However, you can read them over to remind yourself of the expectations. Hopefully, for the majority of the problems, you will know how to start the problems and will not need to actually carry out the details.

1.1: 11, 12, 23, 24
1.2: 7, 9
2.1: 9c, 13, 31
2.2: 2, 9ac, 12ac, 23
2.3: 3, 4, 10, 16, 21ab
2.4: 2 (solve also), 17, 23
2.5: 3, 15, 18, 28a
2.6: 3, 13
2 misc: 1, 9, 17, 22
3.1: 11, 19, 20
3.2: 11, 17, 34
3.3: 11, 21, 23, 24
3.4: 2, 11, 17ab, 18
3.5: 1, 17
3.6: 16, 17
3.7: 7, 10, 29a
3.8: 9, 12
5.2: 2, 5, 7
5.3: 6, 7, 13
5.4: 9, 13, 16, 18
5.5: 6ab, 9
7.5: 2, 5, 16

There are no Chapter 6 problems listed since it is assumed that this material is fresh in your mind. Furthermore, you should be able to write a concise and coherent paragraph on each of the following items:

1. the derivation of the general solution to a first order linear differential equation (see page 36);
2. an explanation for the general solution to differential equations of the form $ay'' + by' + cy = 0$, including the origin of the characteristic equation and the form of the solution based on the roots of the equation (Chapter 3);
3. a statement and proof of the principle of superposition (page 147);
4. a statement and proof of Abel's Theorem (see page 153);
5. the definition of the Laplace transform, the conditions necessary for a function to have a Laplace transform, and the derivation of formulas from this definition.

There will most likely be one or possibly two of these "proofs" on the final exam so plan accordingly. I may allow calculators for the final exam (and I will let you know in advance if you should bring one), but in that case, you must sign the following statement that will appear at the top of the exam:

I certify that I did not use any electronic device for data storage/retrieval or symbolic manipulation.

For example, you may use a calculator to find $\sin 4$ or to solve $e^x - 4x = 12$ or to sketch a graph of a function. However, you may not use your calculator for such things as finding the partial fraction decomposition of $1/s^2(s^2 + 9)$, finding an antiderivative for xe^x, solving the differential equation $y'' - 2y' + 4y = \sin t$, or storing/retrieving proofs or formulas. Using your calculator (or some other electronic device) for these other purposes constitutes academic dishonesty and will be handled appropriately.

Most of the problems on the final exam will be similar to problems that you have seen before. However, it is assumed that you have learned how to approach novel problems during the semester as well and there may be a problem or two on the final exam that forces you to think outside the box. If you have been keeping up this semester and doing the work, you will be well-prepared to tackle any problem that appears on the final exam. If you want some further problems, you may consider the following; some of these (or variations thereof) have appeared on final exams in the past. Cryptic answers are given at the bottom of the page.

1. For problem 3.7.29, find the smallest number T (to four decimal places) so that $|u(t)| < 0.01$ for all $t > T$.

2. Let ϕ be the unique solution to the initial value problem

$$4y'' + y = 4\,\delta(t-4) + 8\,\delta(t-12),\ y(0) = 0,\ y'(0) = 0.$$

   Find the amplitude of ϕ for $t > 12$. Give your answer in symbolic terms if possible and also as a decimal to the nearest thousandth.

3. A certain chemical, let's call it Chemical X, decays at a rate proportional to the square root of the amount present. Initially, there are 400 grams of chemical X; exactly two hours later there are 225 grams. Find the time, to the nearest minute, when there are only 20 grams of Chemical X left.

4. Let $f(t)$ be the solution to the initial value problem $y' = t - y + 2$, $y(0) = \alpha$. Determine the value of α so that the minimum value of $f(t)$ on the interval $[0, \infty)$ is 3.

5. Solve the initial value problem

$$\left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 = 0, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 1.$$

6. Let $y(t)$ be the solution to the initial value problem

$$t^2 y'' - 2y = 0, \quad y(1) = 9, \quad y'(1) = -6.$$

   Find the minimum value of the function y on the interval $t > 0$.

1. The value of T is 39.2495.
2. The amplitude of ϕ for $t > 12$ is $\sqrt{20 + 16\cos 4} \approx 3.089$.
3. There are 20 grams of Chemical X left after just under 6 hours and 13 minutes.
4. The required value of α is $e + 1$.
5. The solution to the initial value problem is $y(t) = \dfrac{2t}{2-t}$.
6. The minimum value of $y(t)$ for $t > 0$ is $3\sqrt[3]{16}$.

Problem: Derive, with careful explanation, the solution to a general first order linear differential equation.

Solution: The general first order linear differential equation has the form $y' + p(t)y = q(t)$, where p and q are continuous functions. The sum on the left, with a little thought, hints at a product rule derivative. Suppose we multiply the equation by some integrating factor $I(t)$:

$$I(t)y' + p(t)I(t)y = I(t)q(t) \qquad \text{and compare it to} \qquad I(t)y' + I'(t)y = I(t)q(t),$$

where the second form contains the derivative of the product $I(t)y$. In order for the two equations to align, we need

$$I'(t) = p(t)I(t) \qquad \text{and thus} \qquad I(t) = Ae^{\int p(t)\,dt},$$

where A is a constant. Taking A to be 1 and using the resulting function as our value for $I(t)$, the differential equation becomes

$$\big(I(t)y\big)' = I(t)q(t) \qquad \text{and it follows that} \qquad I(t)y = \int I(t)q(t)\,dt + C,$$

where C is an arbitrary constant. Therefore, the general solution to the differential equation $y' + p(t)y = q(t)$ is

$$y = e^{-\int p(t)\,dt} \int I(t)q(t)\,dt + Ce^{-\int p(t)\,dt},$$

where C is any constant. ■

Problem: State and prove the principle of superposition for second order differential equations.

Solution: Consider the second order linear homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$. The principle of superposition states the following: if y_1 and y_2 are solutions to the differential equation, then $y = c_1y_1 + c_2y_2$ is also a solution to the differential equation, where c_1 and c_2 are arbitrary constants. To verify this, we use basic properties of derivatives and obtain

$$\begin{aligned}
y'' + p(t)y' + q(t)y &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\
&= c_1y_1'' + c_2y_2'' + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) \\
&= c_1\big(y_1'' + p(t)y_1' + q(t)y_1\big) + c_2\big(y_2'' + p(t)y_2' + q(t)y_2\big) \\
&= c_1 \cdot 0 + c_2 \cdot 0 = 0,
\end{aligned}$$

where the last steps follow from the fact that y_1 and y_2 satisfy the original differential equation. ■

Problem: Explain how to solve the second order differential equation $ay'' + by' + cy = 0$, where a, b, and c are constants.

Solution: Since the derivatives of a function of the form e^{rt}, where r is a constant, all have a similar form, it is reasonable to guess that a solution to the differential equation may involve functions of this type. Suppose that $y = e^{rt}$. We then have

$$0 = ay'' + by' + cy = ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt}.$$

Since e^{rt} is never zero, we find that $ar^2 + br + c = 0$. This quadratic equation is known as the characteristic equation of the differential equation. If r is a root of this equation, then e^{rt} is a solution to the differential equation. However, care must be taken when the roots are repeated or are complex. Since there are three possibilities for the roots of a quadratic equation, the solution to the differential equation has three possible forms.

i. If there are two distinct real roots r_1 and r_2, then $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
ii. If there is a repeated real root r_1, then $y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$.
iii. If the roots are complex numbers of the form $\lambda \pm \mu i$, then $y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$.

In each case, the numbers c_1 and c_2 represent arbitrary constants (see the previous problem). ■

Problem: State and prove Abel's Theorem.

Solution: Abel's Theorem states that if y_1 and y_2 are two solutions to the second order linear differential equation $y'' + p(t)y' + q(t)y = 0$, then there exists a constant C such that $W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt}$, where W represents the Wronskian of the two functions. To prove this, we let $W(t) = W(y_1, y_2)(t)$ to simplify the notation and note that

$$W(t) = y_1 y_2' - y_1' y_2 \qquad \text{and} \qquad W'(t) = (y_1 y_2'' + y_1' y_2') - (y_1' y_2' + y_1'' y_2) = y_1 y_2'' - y_1'' y_2.$$

Since y_1 and y_2 satisfy the differential equation, we know that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \qquad \text{and} \qquad y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiplying the first equation by $-y_2$ and the second equation by y_1, then adding the results yields

$$\big(-y_1'' y_2 - p(t)y_1' y_2\big) + \big(y_1 y_2'' + p(t)y_1 y_2'\big) = 0 \qquad \text{or} \qquad y_1 y_2'' - y_1'' y_2 + p(t)\big(y_1 y_2' - y_1' y_2\big) = 0.$$

Referring to our properties of the Wronskian, the last equation has the form $W'(t) = -p(t)W(t)$. It follows that $W(t) = Ce^{-\int p(t)\,dt}$ for some constant C. ■

Problem: Define the Laplace transform and state conditions under which a function f has a Laplace transform.

Solution: The Laplace transform of a function f defined on the interval $[0, \infty)$ is defined by

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for all values of s for which the improper integral converges. In order for the Laplace transform of f to exist, the function f must be piecewise continuous on each interval $[0, b]$ for $b > 0$ and exponentially bounded, that is, there must exist constants K and a such that $|f(t)| \leq Ke^{at}$. Under these conditions on f, the Laplace transform of f is guaranteed to exist at least for $s > a$.

As a simple example, we will find that Laplace transform of $f(t) = t$. In this case, we find that

$$\begin{aligned}
\mathcal{L}\{t\}(s) &= \int_0^\infty e^{-st} t\, dt && \text{definition of Laplace transform} \\
&= \lim_{T\to\infty} \int_0^T te^{-st}\, dt && \text{definition of improper integral} \\
&= \lim_{T\to\infty} \frac{1}{s^2}(-st-1)e^{-st}\Big|_0^T && \text{formula from integration by parts} \\
&= \lim_{T\to\infty} \frac{1}{s^2}\Big(-\frac{sT+1}{e^{sT}} + 1\Big) && \text{evaluate the integral} \\
&= \frac{1}{s^2}\Big(0+1\Big) && \text{find the limit, assuming } s > 0 \\
&= \frac{1}{s^2} && \text{simplify}
\end{aligned}$$