Total: Name:

**Math 126** **Final Exam** **Fall 2010**

Write neat, concise, and accurate solutions to each of the problems in the space provided—I will not give any credit for steps I cannot follow. Your solutions should be written in the style expected for collected homework problems. Pay particular attention to correct use of notation and use sentences when appropriate. Each of the thirteen problems is worth six points. You may use a calculator for this exam. However, I am assuming that you do not use your calculator or other electronic device for integrating functions (either definite or indefinite), summing infinite series, calling for help, etc; any such use will be treated as cheating.

1. State both versions of the Fundamental Theorem of Calculus.

I. If $f$ is continuous on $[a,b]$ and $F$ is defined by $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$, then $F'(x) = f(x)$ for all $x \in [a,b]$.

II. If $f$ is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$.

2. Suppose that $v(t) = 9t - t^3$ gives the velocity in meters per second of a particle at time $t$ seconds. Find the total distance traveled by the particle from time $t = 0$ to time $t = 4$ seconds.

We need to compute $\int_0^4 |v(t)|\,dt$.

-3 if displacement

8m

$v(t) = t(9-t^2)$ 0 when $t = 0, \pm 3$

$$\int_0^4 |9t - t^3|\,dt = \int_0^3 (9t - t^3)\,dt + \int_3^4 (t^3 - 9t)\,dt$$

$$= \left(\tfrac{9}{2}t^2 - \tfrac{1}{4}t^4\right)\Big|_0^3 + \left(\tfrac{1}{4}t^4 - \tfrac{9}{2}t^2\right)\Big|_3^4$$

$$= \left(\tfrac{81}{2} - \tfrac{81}{4}\right) + \left(\tfrac{175}{4} - \tfrac{63}{2}\right)$$

$$= \tfrac{81}{4} + \tfrac{49}{4}$$

$$= \tfrac{65}{2}$$

The particle travels $32\tfrac{1}{2}$ meters for $0 \le t \le 4$.

3. Find all of the antiderivatives of the function $f(x) = 12x^2 \ln x$.

$\int f(x)\,dx = \int 12x^2 \ln x\,dx$

$u = \ln x \qquad dv = 12x^2\,dx$

$du = \frac{1}{x}\,dx \qquad v = 4x^3$

$= 4x^3 \ln x - \int 4x^2\,dx$ integration by parts

$= 4x^3 \ln x - \frac{4}{3}x^3 + C$

all the antiderivatives of $f(x)$ have this form.

4. Find the sum of the series $\sum_{k=1}^{\infty} \frac{2^k + (-1)^{k+1}}{5^k}$.

$$\sum_{k=1}^{\infty} \frac{2^k + (-1)^{k+1}}{5^k} = \sum_{k=1}^{\infty} \frac{2^k}{5^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{5^k}$$ split up

$$= \sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^k - \sum_{k=1}^{\infty} \left(-\frac{1}{5}\right)^k$$ rewrite

$$= \frac{2/5}{1 - 2/5} - \frac{-1/5}{1 + 1/5}$$ geometric series

$$= \frac{2}{3} + \frac{1}{6}$$ simplify

$$= \frac{5}{6}$$

The sum of the series is $5/6$.

5. Determine the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(4k+1)}(x-5)^k$. Be certain that all of your conclusions are justified.

We use the Ratio Test.

$$L = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(x-5)^{k+1}}{3^{k+1}(4k+5)} \cdot \frac{3^k(4k+1)}{(-1)^k(x-5)^k} \right|$$

$$= \lim_{k\to\infty} \frac{1}{3}\cdot\frac{4k+1}{4k+5}\cdot|x-5| = \frac{|x-5|}{3}.$$

We need $L<1$ so $|x-5|<3$; the radius of convergence is 3.
We must check the endpoints $x=2$ and $x=8$ separately.

$x=2$ gives $\sum_{k=0}^{\infty} \frac{1}{4k+1}$, diverges by comparison with $\sum_{k=1}^{\infty} \frac{1}{k}$

$x=8$ gives $\sum_{k=0}^{\infty} \frac{(-1)^k}{4k+1}$, converges by the alternating series Test

The interval of convergence is thus $(2,8]$.

+ 4 if correct $\rho$ and endpoints

6. Consider the function $f$ defined by $f(x) = \dfrac{e^{x^2} - 1}{x}$. Find the Maclaurin series for this function then use it to find the value of $f^{(101)}(0)$.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$ known fact

$$\frac{e^{x^2} - 1}{x} = \sum_{k=1}^{\infty} \frac{1}{k!} x^{2k-1}$$ $-1$ removes first term; $\div x$ reduces exponent by 1

3

This is the Maclaurin series for $f(x)$.
We know that $f^{(k)}(0) = k!\, c_k$ so $f^{(101)}(0) = 101!\, c_{101}$.
Since $c_{101}$ is the coefficient of $x^{101}$, we consider $k = 51$ and find that $c_{101} = \frac{1}{51!}$. It follows that

3

$$f^{(101)}(0) = \frac{101!}{51!}.$$

7. The sequence of Lucas numbers is $1, 3, 4, 7, 11, 18, 29, \ldots$. These numbers are defined by $\ell_1 = 1$, $\ell_2 = 3$, and $\ell_{n+1} = \ell_n + \ell_{n-1}$ for each $n \geq 2$. Use mathematical induction to prove that

$$\ell_1^2 + \ell_2^2 + \ell_3^2 + \cdots + \ell_n^2 = \ell_n \ell_{n+1} - 2.$$

for each positive integer $n$.

We first check this formula for small values of $n$.

$n = 1$: $1^2 = 1 \cdot 3 - 2$

$n = 2$: $1^2 + 3^2 = 3 \cdot 4 - 2$

$n = 3$: $1^2 + 3^2 + 4^2 = 4 \cdot 7 - 2$

$n = 4$: $1^2 + 3^2 + 4^2 + 7^2 = 7 \cdot 11 - 2$

All of these equations are valid. Suppose that the formula is valid for some positive integer $k$, that is,

$$l_1^2 + l_2^2 + \cdots + l_k^2 = l_k l_{k+1} - 2.$$

We then have

$$\begin{aligned}
l_1^2 + l_2^2 + \cdots + l_{k+1}^2 &= l_1^2 + l_2^2 + \cdots + l_k^2 + l_{k+1}^2 && \text{include a term} \\
&= l_k l_{k+1} - 2 + l_{k+1}^2 && \text{by hypothesis} \\
&= l_{k+1}(l_k + l_{k+1}) - 2 && \text{factor} \\
&= l_{k+1} l_{k+2} - 2 && \text{defn of Lucas number}
\end{aligned}$$

The formula thus holds for $k+1$. By the PMI, the formula

$$l_1^2 + l_2^2 + l_3^2 + \cdots + l_n^2 = l_n l_{n+1} - 2$$

is valid for every positive integer $n$.

8. This problem involves the integral $\int_0^1 \sin(x^2)\,dx$. (Since both parts are asking you to estimate the same value, you should expect your numerical answers to be reasonably close to one another.)

a. Use a Maclaurin series to express this integral as an infinite series, then find the sum of the first four terms (to six decimal places) to approximate the value of the integral.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(2k+1)!}$$

$$\int_0^1 \sin(x^2)\,dx \doteq \frac{1}{3\cdot 1!} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!}$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600}$$

$$= 0.310268$$

This gives the Maclaurin series estimate.

b. Use Simpson's rule with $n = 4$ to approximate the value (to six decimal places) of the integral.

$$\int_0^1 \sin(x^2)\,dx \doteq \frac{1}{12}\left(\sin 0 + 4\sin\tfrac{1}{16} + 2\sin\tfrac{1}{4} + 4\sin\tfrac{9}{16} + \sin 1\right)$$

$$\doteq \frac{1}{12}\left(0 + 0.2498 + 0.4948 + 2.1332 + 0.8415\right)$$

$$\doteq \frac{1}{12}(3.71932687)$$

$$\doteq 0.309944$$

Simpson's rule gives a similar value.

9. Evaluate each of the following integrals.

a. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\sqrt{4-x^2} + C$$

guess and check works well here

b. $\int \frac{x}{4-x^2}\,dx$

$$\int \frac{x}{4-x^2}\,dx = -\frac{1}{2}\ln|4-x^2| + C$$

basic formula, need absolute values

c. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\frac{x}{2} + C$$

an exact basic formula

- 4 if complete square and mess up

10. Evaluate $\displaystyle\int \frac{\overset{3x+2}{\cancel{2x-7}}}{x^2 - 6x - 16}\,dx$.

Since the denominator factors, we will use partial fractions.

$$\frac{3x+2}{(x-8)(x+2)} = \frac{A}{x-8} + \frac{B}{x+2}$$

$$3x+2 = A(x+2) + B(x-8)$$

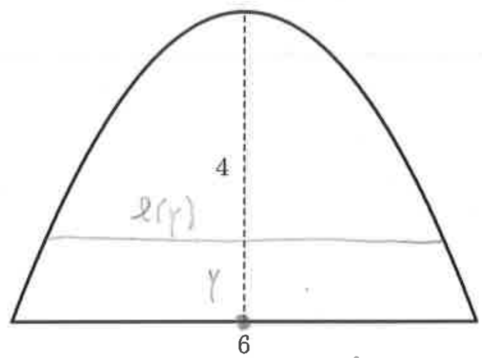
$$x = 8 \implies 26 = 10A \implies A = 13/5$$

$$x = -2 \implies -4 = -10B \implies B = 2/5$$

$$\int \frac{3x+2}{x^2-6x-16}\,dx = \int \left( \frac{13/5}{x-8} + \frac{2/5}{x+2} \right) dx$$

$$= \frac{13}{5} \ln|x-8| + \frac{2}{5} \ln|x+2| + C$$

11. Find the force exerted by a liquid on one side of the vertically submerged parabolic plate shown below. The units on the figure are feet and the top of the plate is five feet beneath the surface of the liquid. Assume that the weight density of the liquid is 60 lb/ft$^3$.




put origin here

$y = 4 - ax^2$

$0 = 4 - a \cdot 3^2$

$a = \frac{4}{9}$

eq of parabola

$y = 4 - \frac{4}{9}x^2$

solve for $x$

$x = \pm\sqrt{\frac{9}{4}(4-y)} = \pm \frac{3}{2}\sqrt{4-y}$

It follows that $\ell(y) = 3\sqrt{4-y}$.

The water level is $y = 9$.

The force on the parabolic plate is

$$F = \int_0^4 60(9-y)\, 3\sqrt{4-y}\, dy$$

let $u = 4 - y$

$$= 180 \int_4^0 (u+5)\, u^{1/2}\, (-du)$$

$$= 180 \int_0^4 (u^{3/2} + 5u^{1/2})\, du$$

$$= 180 \left( \frac{2}{5} u^{5/2} + \frac{10}{3} u^{3/2} \right)$$

$$= 180 \left( \frac{64}{5} + \frac{80}{3} \right)$$

$$= 180 \cdot 16 \left( \frac{4}{5} + \frac{5}{3} \right)$$

$$= 180 \cdot 16 \cdot \frac{12+25}{15}$$

$$= 12 \cdot 16 \cdot 37$$

$$= 7104$$

$$\begin{array}{r} 222 \\ 32 \\ \hline 444 \\ 666 \\ \hline 7104 \end{array}$$

The total force on the plate is 7104 pounds.

12. Determine whether or not each of the following infinite series converges. Give reasons for your responses, including the names of any tests you use.

a. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{3}}$

$\lim_{k\to\infty} \frac{1}{\sqrt[k]{3}} = 1 \neq 0$ so the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{3}}$ diverges by the Divergence Test.

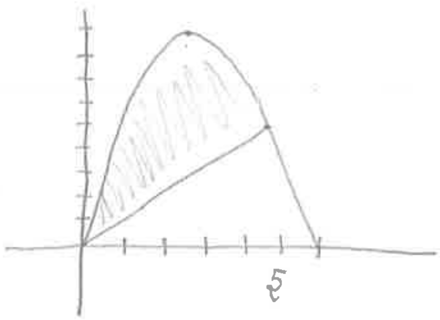
b. $\sum_{k=1}^{\infty} \frac{k}{2^k}$

Since $L = \lim_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{2} = \frac{1}{2} < 1$, the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges by the Root Test.

c. $\sum_{k=1}^{\infty} \frac{1}{5k+1}$

Note that $\frac{1}{5k+1} \geq \frac{1}{5k+k} = \frac{1}{6} \cdot \frac{1}{k}$ for all $k$. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent $p$-series, the series $\sum_{k=1}^{\infty} \frac{1}{5k+1}$ diverges by the Comparison Test.

13. Let $R$ be the region bounded by the curves $y = 6x - x^2$ and $y = x$. For each of the following, write down an expression involving one or more integrals that represents the requested quantity. **DO NOT EVALUATE THE INTEGRALS.**




a) the volume of the solid that is generated when $R$ is revolved around the line $x = 10$

$$V = \int_0^5 2\pi(10-x)(5x-x^2)\,dx$$

$\frac{625}{2}\pi$

b) the volume of the solid whose base is $R$ and each cross-section of the solid taken perpendicular to the $x$-axis is a semicircle

$$V = \int_0^5 \frac{\pi}{8}(5x-x^2)^2\,dx$$

$\frac{625}{48}\pi$

c) the $x$-coördinate of the center of mass of $R$, assuming that the density has a constant value of $\rho$

$$\bar{x} = \frac{\int_0^5 x\rho(5x-x^2)\,dx}{\int_0^5 \rho(5x-x^2)\,dx}$$

$\frac{5}{2}$

$\bar{y} = 5$

BONUS QUESTION: This is not part of the test. If you have extra time and want to continue thinking about mathematics, find the sum of the series $\sum_{k=1}^{\infty} \frac{k^2}{k!}$. A correct answer, along with coherent reasoning, is worth 3 extra points.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

method I $\quad x \cdot \frac{d}{dx}$

$$x e^x = \sum_{k=1}^{\infty} \frac{k}{k!} x^k$$

$$x(x+1)e^x = \sum_{k=1}^{\infty} \frac{k^2}{k!} x^k \quad \text{plug in 1 to get} \quad 2e = \sum_{k=1}^{\infty} \frac{k^2}{k!}$$

method 2

$$\sum_{k=1}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e$$

$$\sum_{k=1}^{\infty} \frac{k^2}{k!} = \sum_{k=1}^{\infty} \frac{k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{k+1}{k!} = \sum_{k=1}^{\infty} \frac{k}{k!} + \sum_{k=0}^{\infty} \frac{1}{k!} = e + e = 2e$$

76
67
68
67
64
63
60
58
55
54, 54
53, 53
50
43

| | |
|---|---|
| 68 - 80 | 3 |
| 56 - 67 | 5 |
| 44 - 55 | 6 |
| 32 - 43 | 1 |
| 0 - 31 | 0 |

very lenient

$\tilde{x} = 58$

Total: Name:

**Math 126B** **Final Exam** **Fall 2010**

Write neat, concise, and accurate solutions to each of the problems in the space provided—I will not give any credit for steps I cannot follow. Your solutions should be written in the style expected for collected homework problems. Pay particular attention to correct use of notation and use sentences when appropriate. Each of the thirteen problems is worth six points. You may use a calculator for this exam. However, I am assuming that you do not use your calculator or other electronic device for integrating functions (either definite or indefinite), summing infinite series, calling for help, etc; any such use will be treated as cheating.

1. State the definition of the derivative and the definition of the integral. Be certain to include all of the appropriate words.

The derivative of a function $f$ is another function $f'$ defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
for all values of $x$ in the domain of $f$ for which the limit exists.

The integral of a continuous function $f$ on an interval $[a,b]$, denoted $\int_a^b f(x)\,dx$, is defined by
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \frac{b-a}{n}.$$

2. Suppose that $v(t) = 9t - t^3$ gives the velocity in meters per second of a particle at time $t$ seconds. Find the total distance traveled by the particle from time $t = 0$ to time $t = 4$ seconds.

distance is $\int_a^b |v(t)|\,dt$

$9t - t^3 = t(9 - t^2) = 0$ for $t = 0, \pm 3$

$$\int_0^4 |9t - t^3|\,dt = \int_0^3 (9t - t^3)\,dt + \int_3^4 (t^3 - 9t)\,dt$$
$$= \left(\frac{9}{2}t^2 - \frac{1}{4}t^4\right)\Big|_0^3 + \left(\frac{1}{4}t^4 - \frac{9}{2}t^2\right)\Big|_3^4$$
$$= \left(\frac{81}{2} - \frac{81}{4}\right) + (64 - 72) - \left(\frac{81}{4} - \frac{81}{2}\right)$$
$$= \frac{81}{4} - 8 + \frac{81}{4}$$
$$= \frac{81}{2} - \frac{16}{2}$$
$$= \frac{65}{2}$$

The particle travels $32\frac{1}{2}$ meters for $0 \le t \le 4$.

3. Find all of the antiderivatives of the function $f(x) = 12x^2 \ln x$.

$\int 12x^2 \ln x \, dx$ gives all the antiderivatives

integration by parts $\quad u = \ln x \quad du = \frac{1}{x}dx \quad dv = 12x^2 dx \quad v = 4x^3$

$$\int 12x^2 \ln x \, dx = 4x^3 \ln x - \int 4x^2 dx$$
$$= 4x^3 \ln x - \frac{4}{3}x^3 + C$$

$$\frac{k^2 \sin^2(3/k)}{1+\cos(3/k)} \to \frac{9}{2}$$

4. Find the limit of the sequence $\left\{k^2(1 - \cos(3/k))\right\}$.

$\infty \cdot 0$ form

We will compute $\lim_{x\to\infty} x^2\left(1-\cos(3/x)\right)$.

$\frac{0}{0}$ forms

$$\lim_{x\to\infty} \frac{1-\cos(3/x)}{1/x^2} = \lim_{x\to\infty} \frac{\sin(3/x)(-3/x^2)}{-2/x^3}$$
$$= \lim_{x\to\infty} \frac{3}{2} \frac{\sin(3/x)}{1/x}$$
$$= \lim_{x\to\infty} \frac{3}{2} \cdot \frac{\cos(3/x)(-3/x^2)}{-1/x^2}$$
$$= \lim_{x\to\infty} \frac{9}{2}\cos(3/x)$$
$$= 9/2$$

$$\lim_{y\to 0^+} \frac{1-\cos 3y}{y^2}$$
$$\lim_{y\to 0^+} \frac{3\sin 3y}{2\cdot y}$$

Using L'Hopital's Rule, we find that the limit of the sequence is $9/2$.

5. Determine the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{2}{5^k(3k+1)}(x-7)^k$. Be certain that all of your conclusions are justified.

We first use the ratio test to find $\rho$.

$$L = \lim_{k\to\infty} \frac{2|x-7|^{k+1}}{5^{k+1}(3k+4)} \cdot \frac{5^k(3k+1)}{2|x-7|^k}$$

$$= \lim_{k\to\infty} \frac{1}{5}\cdot\frac{3k+1}{3k+4}|x-7|$$

$$= \frac{|x-7|}{5}$$

Since we need $L<1$ for convergence, we see that $|x-7|<5$, that is, $\rho = 5$. To check the endpoints 2 and 12, we consider the individual series.

$x=2$ gives $\sum_{k=0}^{\infty} \frac{2(-1)^k}{3k+1}$, which converges by the AST since $\left\{\frac{2}{3k+1}\right\}$ decreases to 0

$x=12$ gives $\sum_{k=0}^{\infty} \frac{2}{3k+1}$, which diverges by comparison to $\sum_{k=1}^{\infty} \frac{1}{k}$, a divergent $p$-series

$$\frac{2}{3k+1} \geq \frac{2}{3k+k} = \frac{1}{2}\cdot\frac{1}{k}$$

The interval of convergence for the power series is $[2, 12)$.

6. Consider the function $f$ defined by $f(x) = \frac{\sin x - x}{x^2}$. Find the Maclaurin series for this function then use it to find the value of $f^{(101)}(0)$.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\frac{\sin x - x}{x^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k-1}$$

This is the Maclaurin series for $f$.

We know $f^{(k)}(0) = k!\, c_k$.

Thus $f^{(101)}(0) = 101!\, c_{101}$

$c_{101}$ is the coefficient of $x^{101}$, which occurs when $k = 51$. Thus $c_{101} = \frac{(-1)^{51}}{103!}$

It follows that $f^{(101)}(0) = -\frac{101!}{103!} = \frac{-1}{102 \cdot 103}$.

7. Use mathematical induction to prove that

$$f_3 + f_6 + f_9 + \cdots + f_{3n} = \frac{f_{3n+2} - 1}{2}$$

for each positive integer $n$, where $f_n$ represents the $n$th Fibonacci number. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

We first check a few values of $n$.

$n = 1$ gives $2 = \frac{5-1}{2}$.

$n = 2$ gives $2 + 8 = \frac{21-1}{2}$.

$n = 3$ gives $2 + 8 + 34 = \frac{89-1}{2}$.

So the formula is valid for $n = 1, 2, 3$. Suppose that the formula is valid for some positive integer $k$. We then have

$$f_3 + f_6 + f_9 + \cdots + f_{3k} + f_{3k+3} = \frac{f_{3k+2} - 1}{2} + f_{3k+3}$$

$$= \frac{f_{3k+3} + f_{3k+3} + f_{3k+2} - 1}{2}$$

$$= \frac{f_{3k+3} + f_{3k+4} - 1}{2}$$

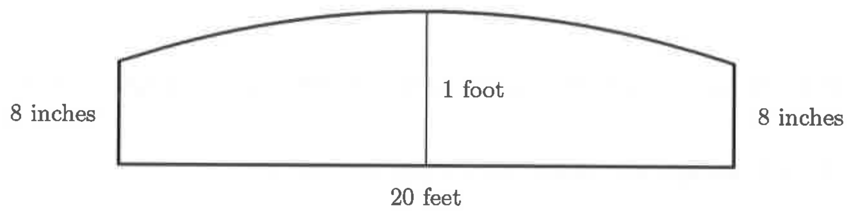
$$= \frac{f_{3k+5} - 1}{2},$$

which shows that the formula is true for $k+1$. By the PMI, we know that

$$f_3 + f_6 + f_9 + \cdots + f_{3n} = \frac{f_{3n+2} - 1}{2}$$

is valid for each $n \in \mathbb{Z}^+$.

8. Determine (to the nearest cubic yard) the number of cubic yards of crushed rock necessary to make a roadbed one mile long with cross section (not to scale) shown below.




Assume that the crown of the roadbed is a parabola. (There are 5280 feet in a mile.)

There are several options here. If we put the origin of a coordinate system at the middle of the base, then the equation of the parabola is

$y = 1 - ax^2$, where $\frac{2}{3} = 1 - a \cdot 10^2$ or $a = \frac{1}{300}$.

The area of the cross-section is thus

$$\int_{-10}^{10} \left(1 - \frac{1}{300}x^2\right) dx = 2\left(x - \frac{1}{900}x^3\right)\Big|_0^{10}$$

$$= 2\left(10 - \frac{10}{9}\right) = \frac{160}{9}$$

The units here are square feet. The total volume of the roadbed is thus

$$\frac{1}{27} \cdot \frac{160}{9} \cdot 5280 = \frac{16 \cdot 176}{81} \cdot 100 \doteq 3476.54321 \text{ yds}^3.$$

The roadbed will require approximately 3477 cubic yards of crushed rock.

9. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x}{x^2+9}\,dx$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln(x^2+9) + C$$

guess and check

b. $\displaystyle\int \frac{1}{x^2+9}\,dx$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

basic formula

c. $\displaystyle\int \frac{1}{x^2-9}\,dx$

$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\int\left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

simple partial fractions

include reduction formula for cosine

10. Evaluate $\int \frac{1}{(x^2+4)^2}\,dx$.

We will use trig substitution.

let $x = 2\tan\theta$

then $dx = 2\sec^2\theta\,d\theta$

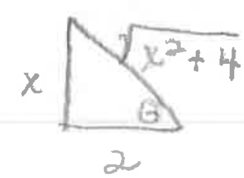
$x^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$

$$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{2\sec^2\theta}{16\sec^4\theta}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{8}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right) + C$$

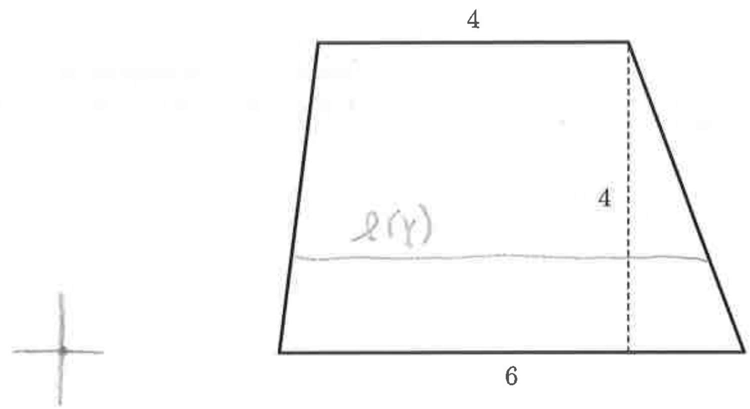
return to $x$ using a triangle




$$\int \frac{1}{(x^2+4)^2}\,dx = \frac{1}{16}\left(\frac{2}{\sqrt{x^2+4}}\cdot\frac{x}{\sqrt{x^2+4}} + \arctan\left(\frac{x}{2}\right)\right) + C$$

$$= \frac{1}{16}\left(\frac{2x}{x^2+4} + \arctan\left(\frac{x}{2}\right)\right) + C$$

11. Find the force exerted by a liquid on one side of the vertically submerged trapezoidal plate shown below. The units on the figure are feet and the top of the plate is four feet beneath the surface of the liquid. Assume that the weight density of the liquid is 60 lb/ft$^3$.




With the origin as indicated, the horizontal distance $\ell(y)$ across the trapezoid is given by $\ell(y) = 6 - \frac{1}{2}y$ for $0 \le y \le 4$. The force on the plate is thus

$$
\begin{aligned}
F &= \int_0^4 60(8-y)(6-\tfrac{1}{2}y)\,dy \\
&= 20\int_0^4 (8-y)(12-y)\,dy \\
&= 20\int_0^4 (y^2-20y+96)\,dy \\
&= 20\left(\tfrac{1}{3}y^3 - 10y^2 + 96y\right)\Big|_0^4 \\
&= 20\left(\tfrac{64}{3} - 160 + 384\right) \\
&= 20\left(\tfrac{64}{3} + 224\right) \\
&= 10(64 + 672) \\
&= 7360
\end{aligned}
$$

The force exerted by the liquid on the plate is 7360 pounds.

12. Determine whether or not each of the following infinite series converges. Give reasons for your responses, including details and the names of any tests you use.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{5}}$

Since $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt[k]{5}} = 1 \neq 0$, the series diverges by the Divergence Test.

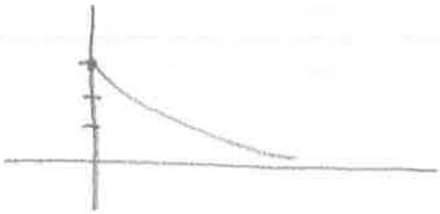
b. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{3^k}$

Since $\displaystyle\lim_{k\to\infty} \sqrt[k]{\frac{k}{3^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{3} = \frac{1}{3} < 1$, the series converges by the Root Test.

c. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{7k+1}$

Since $\displaystyle\lim_{k\to\infty} \frac{\frac{1}{7k+1}}{1/k} = \lim_{k\to\infty} \frac{k}{7k+1} = \frac{1}{7} > 0$ and the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent $p$-series, the given series diverges by the Limit Comparison Test.

13. Let $R$ be the region that lies below the curve $y = \dfrac{2+e^x}{e^{2x}}$ and above the $x$-axis on the interval $[0, \infty)$. Find the volume of the solid that is generated when $R$ is revolved around the $x$-axis.



$y = 2e^{-2x} + e^{-x}$

$\frac{dy}{dx} = -4e^{-2x} - e^{-x}$

decreasing on $[0, \infty)$

$$V = \int_0^\infty \pi \left(\frac{2+e^x}{e^{2x}}\right)^2 dx$$

$$= \pi \int_0^\infty \frac{4 + 4e^x + e^{2x}}{e^{4x}}\, dx$$

$$= \pi \lim_{b\to\infty} \int_0^b \left(4e^{-4x} + 4e^{-3x} + e^{-2x}\right) dx$$

$$= \pi \lim_{b\to\infty} \left(-e^{-4x} - \tfrac{4}{3}e^{-3x} - \tfrac{1}{2}e^{-2x}\right)\Big|_0^b$$

$$= \pi \lim_{b\to\infty} \left[\left(-e^{-4b} - \tfrac{4}{3}e^{-3b} - \tfrac{1}{2}e^{-2b}\right) - \left(-1 - \tfrac{4}{3} - \tfrac{1}{2}\right)\right]$$

$$= \pi\left(\tfrac{3}{2} + \tfrac{4}{3}\right)$$

$$= \frac{17\pi}{6}$$

The volume of the solid is $\frac{17\pi}{6}$ cubic units.

BONUS QUESTION: This is not part of the test. If you have extra time and want to continue thinking about mathematics, find the sum of the series $\sum_{k=1}^{\infty} \frac{(2+(-1)^k)^k}{5^k}$. A correct answer, along with coherent reasoning, is worth 3 extra points.

$$\sum_{k=1}^{\infty} \frac{(2+(-1)^k)^k}{5^k} = \underbrace{\sum_{k=1}^{\infty} \frac{1}{5^{2k-1}}}_{\text{odd terms}} + \underbrace{\sum_{k=1}^{\infty} \frac{3^{2k}}{5^{2k}}}_{\text{even terms}}$$

$$= \frac{1/5}{1-\frac{1}{25}} + \frac{9/25}{1-\frac{9}{25}}$$

$$= \frac{5}{24} + \frac{9}{16}$$

$$= \frac{10+27}{48}$$

$$= \frac{37}{48}$$

77
73
66
65
63
59
57
56
54
51, 51
50
46, 46
45
44
36
27

68
56
44
32
0

$\tilde{x} = 52.5$