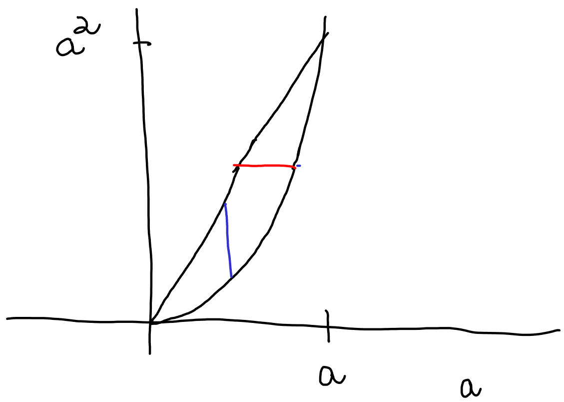
Name: due September 28 by 3:00 pm Pacific

Math 126 **Homework Assignment 13** **Fall 2020**

1. Find the center of mass of the region bounded by $y = x^2$ and $y = ax$, where a is a positive constant.



$(ax - x^2)\,dx$

$\left(\sqrt{y} - \frac{y}{a}\right) dy$

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^a x\rho(ax-x^2)\,dx}{\int_0^a \rho(ax-x^2)\,dx} = \frac{\int_0^a (ax^2 - x^3)\,dx}{\int_0^a (ax-x^2)\,dx}$$

$$= \frac{\left(\frac{1}{3}ax^3 - \frac{1}{4}x^4\right)\Big|_0^a}{\left(\frac{1}{2}ax^2 - \frac{1}{3}x^3\right)\Big|_0^a} = \frac{\left(\frac{1}{3} - \frac{1}{4}\right)a^4}{\left(\frac{1}{2} - \frac{1}{3}\right)a^3} = \frac{1/12}{1/6}\,a = \frac{1}{2}a$$

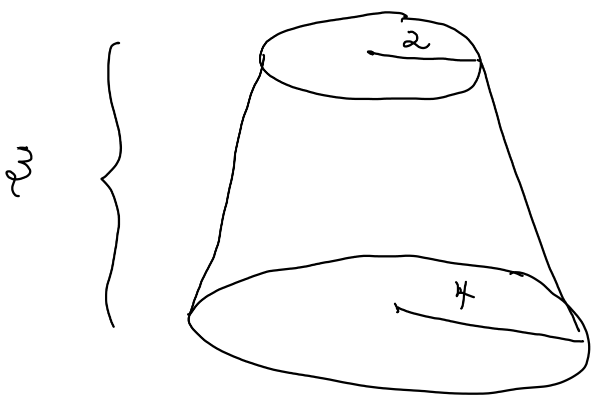
reasonable answer given the shape

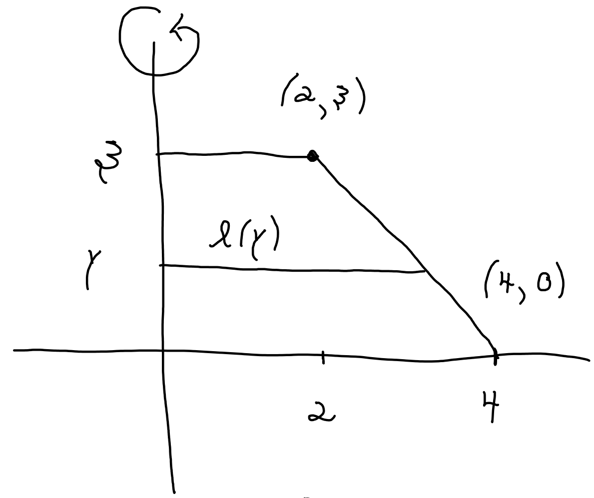
$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_0^{a^2} y\rho\left(\sqrt{y} - \frac{y}{a}\right)dy}{\int_0^{a^2} \rho\left(\sqrt{y} - \frac{y}{a}\right)dy} = \frac{\int_0^{a^2}\left(y^{3/2} - \frac{1}{a}y^2\right)dy}{\int_0^{a^2}\left(y^{1/2} - \frac{1}{a}y\right)dy}$$

$$= \frac{\left(\frac{2}{5}y^{5/2} - \frac{1}{3a}y^3\right)\Big|_0^{a^2}}{\left(\frac{2}{3}y^{3/2} - \frac{1}{2a}y^2\right)\Big|_0^{a^2}} = \frac{\frac{2}{5}a^5 - \frac{1}{3}a^5}{\frac{2}{3}a^3 - \frac{1}{2}a^3} = \frac{\frac{1}{15}}{1/6}\,a^2 = \frac{2}{5}a^2$$

The center of mass of the region is $\left(\frac{1}{2}a, \frac{2}{5}a^2\right)$.

2. A stump with constant density and the shape of a chopped off cone has a top radius of two feet, a bottom radius of four feet, and a height of three feet. How far above the ground is its center of mass?





cross-sections are circles

$$dm = \rho \cdot \pi (\ell(y))^2 \, dy$$

$$y - 3 = \frac{0-3}{4-2}(x-2)$$

$$y - 3 = -\frac{3}{2}(x-2) = -\frac{3}{2}x + 3$$

$$\frac{3}{2}x = 6 - y \quad \text{or} \quad x = 4 - \frac{2}{3}y$$

$$\bar{y} = \frac{\int_0^3 y\rho\pi\left(4-\frac{2}{3}y\right)^2 dy}{\int_0^3 \rho\pi\left(4-\frac{2}{3}y\right)^2 dy} = \frac{\int_0^3 y(6-y)^2\,dy}{\int_0^3 (6-y)^2\,dy}$$

$$= \frac{\int_0^3 (y^3 - 12y^2 + 36y)\,dy}{\int_0^3 (y^2 - 12y + 36)\,dy} = \frac{\left(\frac{1}{4}y^4 - 4y^3 + 18y^2\right)\Big|_0^3}{\left(\frac{1}{3}y^3 - 6y^2 + 36y\right)\Big|_0^3}$$

$$= \frac{3^3\left(\frac{3}{4} - 4 + 6\right)}{3^3\left(\frac{1}{3} - 2 + 4\right)} = \frac{11/4}{7/3} = \frac{33}{28}$$

The center of mass of the stump is $\frac{33}{28}$ feet above the ground.