Name: due October 1

Math 126 **Homework Assignment 11** **Fall 2021**

1. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for each positive integer n.

start with some checking

$n=1 \quad 1^2 = 1\cdot 1$

$n=2 \quad 1^2 + 1^2 = 1\cdot 2$

$n=3 \quad 1^2+1^2+2^2 = 2\cdot 3$

$$n=4 \quad f_1^2 + f_2^2 + f_3^2 + f_4^2 = 1^2+1^2+2^2+3^2 = 15 = 3\cdot 5 = f_4 f_5$$

set S method (not required)

Let S be the set of all positive integers n such that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$. Since $\sum_{i=1}^{1} f_i^2 = 1^2 = 1\cdot 1 = f_1 f_2$, we see that $1 \in S$. Now suppose that $k \in S$ for some positive integer k. This means that $\sum_{i=1}^{k} f_i^2 = f_k f_{k+1}$. We then have

$$\sum_{i=1}^{k+1} f_i^2 = \sum_{i=1}^{k} f_i^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2$$

$$= f_{k+1}(f_k + f_{k+1}) = f_{k+1} f_{k+2},$$

showing that $k+1 \in S$. We have thus proved "if $k \in S$, then $k+1 \in S$". By the PMI, we know $S = \mathbb{Z}^+$. Hence, the equation

$$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$$

is valid for all positive integers n.

2. Prove that $f_1f_2 + f_2f_3 + f_3f_4 + \cdots + f_{2n-1}f_{2n} = f_{2n}^2$ for each positive integer n. (A comment concerning this problem was given in class; note that the pattern ends with an even subscript.)

start with some checking

$n=1$, $f_1 f_2 = 1\cdot 1 = 1^2 = f_2^2$

$n=2$, $\underline{f_1 f_2 + f_2 f_3 + f_3 f_4} = 1\cdot 1 + 1\cdot 2 + 2\cdot 3 = 9 = 3^2 = f_4^2$

$n=3$, $\_\_\_\_ + f_4 f_5 + f_5 f_6 = 9 + 3\cdot 5 + 5\cdot 8 = 64 = 8^2 = f_6^2$

We will use the PMI. As indicated above, the equation is valid for $n = 1, 2, 3$. Suppose that $\sum_{i=1}^{2k-1} f_i f_{i+1} = f_{2k}^2$ for some $k \in \mathbb{Z}^+$. Then

$$\begin{aligned}
\sum_{i=1}^{2k+1} f_i f_{i+1} &= \sum_{i=1}^{2k-1} f_i f_{i+1} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} \\
&= f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} \\
&= f_{2k+2}(f_{2k} + f_{2k+1}) \\
&= f_{2k+2}^2,
\end{aligned}$$

so the equation is valid for $k+1$. By the PMI, the equation $\sum_{i=1}^{2n-1} f_i f_{i+1} = f_{2n}^2$ is true for all $n \in \mathbb{Z}^+$.

This is the more common style for induction arguments, but you may use either style.