Name: due October 21 by 3:00 pm Pacific

Math 126 **Homework Assignment 20** **Fall 2020**

1. For each positive integer n, let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$. Prove that the sequence $\{x_n\}$ converges. (To prove that the sequence is bounded, consider using some over and under estimates for the terms in the sum. As a start, you should write out the first four terms of the sequence.)

$x_1 = \frac{1}{2}$

$x_2 = \frac{1}{3} + \frac{1}{4}$

$x_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

$x_4 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$

Since $x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$ for all n, the sequence $\{x_n\}$ is increasing.

Since $0 < x_n < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1$ for all n, the sequence $\{x_n\}$ is bounded.

By the Completeness Axiom, the sequence $\{x_n\}$ converges.

notes x_n is the sum of n numbers, all less than $1/n$

could also use $\leq \frac{1}{n+1}$ and get $\frac{n}{n+1} < 1$ for all n

$x_{n+1} - x_n$ is a reasonable method to prove increasing since lots of cancellation occurs.

For the record, this sequence appears in Exercise 2c in Section 2.3. You can the find the limit of the sequence.

$\frac{1}{2n+2}$ is smaller than $\frac{1}{2n+1}$ since the denominator is larger

2. Let $r_1 = 6$ and $r_{n+1} = \frac{r_n}{2} + \frac{7}{r_n}$ for each $n \geq 1$. Suppose that we already know that the sequence $\{r_n\}$ converges. Under this assumption, find the limit of the sequence.

Let L be the limit of the sequence $\{r_n\}$. Since

$$\lim_{n\to\infty} r_{n+1} = L \text{ and } \lim_{n\to\infty}\left(\frac{r_n}{2} + \frac{7}{r_n}\right) = \frac{L}{2} + \frac{7}{L},$$

it follows that

$$L = \frac{L}{2} + \frac{7}{L} \iff \frac{L}{2} = \frac{7}{L} \iff L^2 = 14 \iff L = \pm\sqrt{14}.$$

Since $r_n > 0$ for all n, the limit of the sequence is $\sqrt{14}$.

3. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = 3 - (1/a_n)$ for $n \geq 1$. We have already proved that $1 \leq a_n \leq 3$ for all n (see Exercise 3.1.5). Using similar ideas, use math induction to prove that $\{a_n\}$ is an increasing sequence. Then conclude that $\{a_n\}$ converges and find its limit.

$$a_1 = 1$$
$$a_2 = 3 - \frac{1}{1} = 2$$
$$a_3 = 3 - \frac{1}{2} = \frac{5}{2}$$
$$a_4 = 3 - \frac{1}{5/2} = \frac{13}{5}$$

We know $\{a_n\}$ is bounded since $1 \leq a_n \leq 3$ for all n.

We will use the PMI to prove that $a_n < a_{n+1}$ for all positive integers n. Since $a_1 = 1 < 2 = a_2$, the inequality is true when $n = 1$. Now suppose that $a_k < a_{k+1}$ for some positive integer k. Then

$$a_k < a_{k+1} \implies \frac{1}{a_k} > \frac{1}{a_{k+1}} \implies -\frac{1}{a_k} < -\frac{1}{a_{k+1}}$$
$$\implies 3 - \frac{1}{a_k} < 3 - \frac{1}{a_{k+1}} \implies a_{k+1} < a_{k+2},$$

so the inequality is true for $k+1$ as well. By the PMI, we have $a_n < a_{n+1}$ for all n. Thus, the sequence $\{a_n\}$ is increasing. By the Completeness Axiom, the sequence $\{a_n\}$ converges. Let L be the limit of the sequence and note that L is between 2 and 3. The equation $a_{n+1} = 3 - \frac{1}{a_n}$ tells us that

$$L = 3 - \frac{1}{L} \implies L^2 - 3L + 1 = 0 \implies L = \frac{3 \pm \sqrt{9-4}}{2}.$$

Since $L > 2$, it follows that $L = \frac{3+\sqrt{5}}{2}$. Thus the limit of the sequence $\{a_n\}$ is $\frac{3+\sqrt{5}}{2}$.