Name: due October 26 by 3:00 pm Pacific

Math 126 **Homework Assignment 22** **Fall 2020**

1. Use the Integral Test to show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges. Use correct notation for improper integrals and note that the lower limit of integration is 2.

For $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$, we have $f(x) = \frac{1}{x \ln x}$. Since

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1/x}{\ln x}\,dx = \lim_{b\to\infty} \ln|\ln x|\Big|_2^b$$

$\int \frac{du}{u}$ form so $\ln|u|$

$$= \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 2)\right) = \infty,$$

the series diverges by the Integral Test.

For $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$, we have $f(x) = \frac{1}{x(\ln x)^2}$. Since

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1/x}{(\ln x)^2}\,dx = \lim_{b\to\infty} \frac{-1}{\ln x}\Big|_2^b$$

$\int \frac{du}{u^2}$ form so $-\frac{1}{u}$

$$= \lim_{b\to\infty} \left(-\frac{1}{\ln b} - \left(\frac{-1}{\ln 2}\right)\right) = \frac{1}{\ln 2},$$

the series converges by the Integral Test.

2. Find all values of a, where a is a real number, for which the series $\sum_{k=1}^{\infty} \frac{1}{k^{6a-a^2}}$ converges.

From the p-series result, we know that this series converges when $6a - a^2 > 1$. Solving the inequality, we find that

complete the square

$$6a - a^2 > 1 \implies a^2 - 6a + 1 < 0 \implies (a-3)^2 < 8$$

$\sqrt{x^2} = |x|$ $\sqrt{8} = \sqrt{4}\cdot\sqrt{2}$ property of absolute value

$$\implies |a-3| < 2\sqrt{2} \implies -2\sqrt{2} < a-3 < 2\sqrt{2}$$

$$\implies 3 - 2\sqrt{2} < a < 3 + 2\sqrt{2}$$

The series converges for all a that satisfy

$3 - 2\sqrt{2} < a < 3 + 2\sqrt{2}$.

3. For each positive integer n, let $z_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \int_1^n \frac{dx}{x}$. Use the ideas in this section to prove that $\{z_n\}$ is a decreasing sequence of positive terms and thus convergent. For the decreasing part, it is best to consider $z_n - z_{n+1}$ since the terms of the sequence involve additions. To show that all of the terms are positive, look carefully at the inequalities next to the graph in the section.

In the section, we proved that

$$\int_1^n f(x)\,dx < \sum_{k=1}^n a_k < a_1 + \int_1^n f(x)\,dx$$

where $f(x)$ is a decreasing function. When $a_k = \frac{1}{k}$, this inequality becomes

$$\int_1^n \frac{1}{x}\,dx < \sum_{k=1}^n \frac{1}{k} \leq 1 + \int_1^n \frac{1}{x}\,dx$$

get = when $n = 1$

$$\Rightarrow \quad 0 < \sum_{k=1}^n \frac{1}{k} - \int_1^n \frac{1}{x}\,dx \leq 1$$

note the formula for the z_n values

This shows that $0 < z_n \leq 1$ for all n. Hence, the sequence $\{z_n\}$ is bounded. We also have

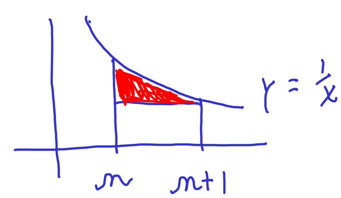
$$z_n - z_{n+1} = \left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \int_1^n \tfrac{1}{x}\,dx\right) - \left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} + \tfrac{1}{n+1} - \int_1^{n+1} \tfrac{1}{x}\,dx\right)$$

$$= -\int_1^n \frac{1}{x}\,dx - \frac{1}{n+1} + \int_1^{n+1} \frac{1}{x}\,dx$$

$$\int_1^{n+1} f(x)\,dx - \int_1^n f(x)\,dx = \int_n^{n+1} f(x)\,dx$$

$$= \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \quad \text{(red area in graph)}$$

$$> 0$$



for all n. This shows that $z_n > z_{n+1}$ for all n so the sequence $\{z_n\}$ is decreasing. Since $\{z_n\}$ is bounded and monotone, it converges by the Completeness Axiom.