Name: due October 30 by 3:00 pm Pacific

Math 126 **Homework Assignment 24** **Fall 2020**

1. Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^2}{2k^4+13}$ is absolutely convergent.

Since $\left|\frac{(-1)^{k+1}k^2}{2k^4+13}\right| = \frac{k^2}{2k^4+13} < \frac{k^2}{2k^4} = \frac{1}{2k^2} < \frac{1}{k^2}$

for all $k \geq 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series,

the series $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}k^2}{2k^4+13}\right|$ converges by the Comparison Test.

Hence, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^2}{2k^4+13}$ is absolutely convergent.

2. Determine (with proof and/or explanation) if the series $\sum_{k=1}^{\infty} \frac{(-1)^k(k+1)}{2k^3+7k^2-1}$ is absolutely convergent, conditionally convergent, or divergent.

We will use the Limit Comparison Test to prove that

$\sum_{k=1}^{\infty} \frac{k+1}{2k^3+7k^2-1}$ converges; it then follows that the series

$\sum_{k=1}^{\infty} \frac{(-1)^k(k+1)}{2k^3+7k^2-1}$ is absolutely convergent.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and

$$\alpha = \lim_{k\to\infty} \frac{\frac{k+1}{2k^3+7k^2-1}}{1/k^2} = \lim_{k\to\infty} \frac{k^3+k^2}{2k^3+7k^2-1} = \frac{1}{2},$$

the series $\sum_{k=1}^{\infty} \frac{k+1}{2k^3+7k^2-1}$ converges by the LCT.

3. Give an example of a series for which $\sum\limits_{k=1}^{\infty} a_k$ converges but $\sum\limits_{k=1}^{\infty} a_k^2$ diverges. (Note that the result of Exercise 5 in the textbook is relevant here.)

Let $a_k = \dfrac{(-1)^k}{\sqrt{k}}$.

$\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k}}$ converges by the AST.

$\sum\limits_{k=1}^{\infty} a_k^2 = \sum\limits_{k=1}^{\infty} \left(\dfrac{(-1)^k}{\sqrt{k}}\right)^2 = \sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ diverges (p-series with $p = 1$).

4. Carefully prove that the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}k}{3k^2+2}$ is conditionally convergent. (Note that two proofs are required; one to show a series diverges and another to show a series converges.)

We first consider the series $\sum\limits_{k=1}^{\infty} \dfrac{k}{3k^2+2}$.

Since $\dfrac{k}{3k^2+2} > \dfrac{k}{4k^2} = \dfrac{1}{4} \cdot \dfrac{1}{k}$ for all $k \geq 2$ and the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{4} \cdot \dfrac{1}{k}$ diverges, the series $\sum\limits_{k=1}^{\infty} \dfrac{k}{3k^2+2}$ diverges by the Comparison Test.

We next look at the sequence $\left\{ \dfrac{k}{3k^2+2} \right\}$. It does converge to 0. To show it is decreasing, let $f(x) = \dfrac{x}{3x^2+2}$ and compute $f'(x) = \dfrac{3x^2+2-6x^2}{(3x^2+2)^2} = \dfrac{2-3x^2}{(3x^2+2)^2}$. Since $f'(x) < 0$ for all $x \geq 1$, the sequence $\left\{ \dfrac{k}{3k^2+2} \right\}$ is decreasing. By the Alternating Series Test, the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1} k}{3k^2+2}$ converges.

With these two results, we see that $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k k}{3k^2+2}$ is conditionally convergent.