Math 287: Independent Study in Geometry

As far as mathematics is concerned, it is difficult to find a subject area with a longer history than geometry. In addition to the content itself, some of which is applicable in everyday life, many students over the centuries have struggled through geometry in an effort to learn how to think logically. In later life, most people remember a few of the theorems and ideas in geometry, but this is much less true today than it was for educated people a century ago. (A brief look at older textbooks in geometry, algebra, and calculus will convince you of this fact.) As you may recall, this basic geometry is often referred to as Euclidean geometry since the Greek mathematician Euclid wrote a textbook known as *The Elements* that has been in use in various ways for over 2000 years. Euclidean geometry begins with a few principles, which most people agree must be true, then proceeds to deduce propositions that follow from these initial truths. If there is such a thing as truth, it would seem to be found in geometry. However, even here doubts and subtleties creep in and force us to rethink our axioms and conclusions. The entire subject area has intrigued many great minds over the past thirty centuries.

In this course, we will (i) spend some time reviewing and extending the ideas presented in a high school geometry class, (ii) work through Books I and II of *The Elements*, and (iii) take a gentle excursion into the realm of hyperbolic geometry, one of the branches of non-Euclidean geometry. The texts required for each portion of the course are (sufficient pages of (iii) are available on the website for this course)

(i) *Geometry for College Students* by I. Martin Isaacs, used copies are readily available via a Google search or see the site `https://bookstore.ams.org/view?ProductCode=AMSTEXT/8`.

(ii) *The Thirteen Books of the Elements, Vol. 1* by Thomas L. Heath, available through Dover Publications.

(iii) *Introduction to Non-Euclidean Geometry* by Harold E. Wolfe, available through Dover Publications.

There are other ways to purchase these texts; you can try an online search. You should be able to obtain all three of them for less than \$100. Be certain that you acquire them in plenty of time for your studies.

In the pages to follow, you will find 36 assignments (plus a short introductory reading assignment). You should plan on roughly three to four hours for each one, but this may vary somewhat depending on the topic. As this is an independent study course, you are expected to learn the material on your own. In fact, this is one of the more important aspects of the course. In addition to learning some useful and important content, you have an opportunity to develop some habits of mind that will help you become an independent learner. These habits include discipline, concentration, perseverance, and patience.

The grading for the course will be based on a journal (40%), a two-hour written midterm exam (30%), and a one-hour oral final exam (30%). The journal should include exercise solutions and comments you want to record about the content as you study the textbooks. The journal should be neat and well-organized and written so that someone else taking the course can pick it up and make sense of what you have been doing. The midterm exam will be scheduled roughly seven weeks into the semester and focus on the material from the Isaacs text. The final exam will be scheduled near or during finals week and will focus on the material from the Euclid and Wolfe texts. Details concerning these exams can be found in the syllabus.

Unlike other math courses you have taken (such as calculus or linear algebra), the theorems and problems in geometry (taken as a whole) do not build up in a nice logical pattern. (This sort of hierarchical pattern, however, does appear on a small scale and it is rather beautiful to see the connections.) The theorems and results are more of a tapestry than a tree or a mountain peak; there are many connections between the results and many different ways to get to the results. When you finish this course (taking a hiking analogy), you will have seen some interesting vistas, learned how to negotiate some difficult terrain, and discovered a few trailheads. Although some trails will be more traveled than others, you will not find the "correct one" or the "best one," but you will have experienced some (just a tiny portion) of the vastness of the subject area of geometry.

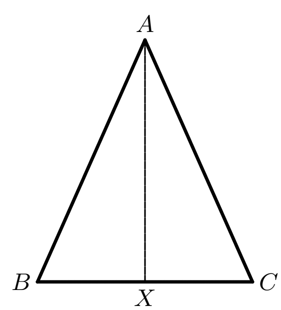
Assignment 0: Read the preface and Section 1A in Isaacs.

1. Do read this material carefully to get a sense for the goals of the author.
2. Referring to Figure 1.1, draw a similar picture for which $BX > XC$. Also, draw a triangle ABC in which the altitude from A does not meet segment BC of the triangle. These figures should be recorded in your journal. You might want to include a few brief thoughts describing your reactions to the reading.
3. We will look briefly at Hilbert's axioms later in the course as well as the parallel postulate and its impact on the study of geometry. In particular, we will prove some of the theorems in hyperbolic geometry.
4. Consider the surface of a sphere, say a smooth basketball, and let lines on this surface be arcs of circles that have the same center as the center of the sphere. You should be able to determine how to draw a "triangle" on this surface with three right angles.

Assignment 1: Read Section 1B in Isaacs.

1. Hopefully, you will find that this section reads rather quickly and discusses some topics familiar to you from your high school geometry course.
2. Since the exercises at the end of this section are elementary, you should do all of them. Note that for these problems you are not allowed to use the Pythagorean Theorem or the fact that the sum of the angles of a triangle is 180°. How does the proof for 1B.6 change if the altitudes are outside of the triangle? Try to do problem 1B.7 two ways; one using HA and another using AAS.
3. Since you are just starting out, we present some sample solutions to make sure you get off on the right foot. We begin with exercise 1B.2; you should think about the exercise before reading the solution.

1B.2: If the altitude from vertex A in triangle $\triangle ABC$ is also the bisector of $\angle A$, then $\triangle ABC$ is isosceles.



Here is one possible proof of this fact using a style often seen in high school geometry classes. You may recall that the abbreviation CPCTE stands for "corresponding parts of congruent triangles are equal."

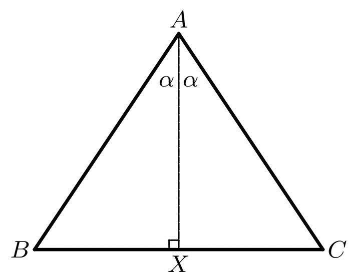
Statement	Reason
1. $\triangle ABC$ with altitude AX	1. given
2. $AX \perp BC$	2. AX is an altitude
3. AX bisects $\angle BAC$	3. given
4. $\angle BAX = \angle CAX$	4. definition of angle bisector
5. $\angle AXB = \angle AXC$	5. both are right angles
6. $AX = AX$	6. obvious
7. $\triangle AXB \cong \triangle AXC$	7. ASA congruence criterion
8. $AB = AC$	8. CPCTE
9. $\triangle ABC$ is isosceles	9. definition of isosceles triangle

On page 8, Isaacs mentions that he does not recommend this style of proof. However, it does have a few advantages since it clearly provides reasons for each step and puts the statements in an organized format. There are times when such a proof, especially in your notes, may be the best option. A more verbal proof in the style of the text is presented below.

Proof: In the figure (see the previous page), we have drawn triangle $\triangle ABC$ with altitude AX. By hypothesis, altitude AX bisects $\angle BAC$. This means that $\angle BAX = \angle CAX$. We also know that $\angle AXB = \angle AXC$ since these are right angles. Hence, triangles $\triangle AXB$ and $\triangle AXC$ are congruent by ASA. It follows that $AB = AC$ and thus $\triangle ABC$ is isosceles. ∎

We offer one more option for a solution to this simple exercise. It uses fewer letters and, at least for some people, is easier to follow.

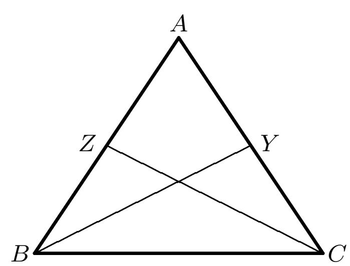
Proof: The exercise is depicted in the figure below. The angles labeled α are equal since altitude AX bisects $\angle A$. Thus $\triangle AXB \cong \triangle AXC$ by ASA. It follows that $AB = AC$ and $\triangle ABC$ is isosceles. ∎



Finally, we present one solution to exercise 1B.4.

1B.4: The medians from the base angles of an isosceles triangle are equal.

Proof: In the figure, $\triangle ABC$ is isosceles with base BC and segments BY and CZ are medians.



Since Y and Z are the midpoints of the sides of the isosceles triangle, we know that

$$BZ = \frac{AB}{2} = \frac{AC}{2} = CY.$$

We also know that $\angle BCY = \angle CBZ$ since the base angles of an isosceles triangle are equal. Now $\triangle BCY$ is congruent to $\triangle CBZ$ by the SAS congruence criterion. It follows that $BY = CZ$. (Note that we have used the notation AC to represent both the line segment and the length of the line segment. This is technically not correct, but it should rarely cause any confusion.) ∎

4. In your journal, write two proofs that provide solutions to exercise 1B.8, one in the statement/reason format and another in paragraph form.

Assignment 2: Read Sections 1C and 1D in Isaacs.

1. Except for the basic idea, you may skim Problem 1.5 if you find it confusing or uninteresting.
2. Do all of the exercises at the end of Section D. You should find that the first ten exercises go pretty quickly, usually requiring little more than a picture and a triangle congruence observation. You may use earlier results to help prove later results. For instance, you might find exercise 6 useful in the solution of exercise 10. By the way, note that two proofs are required for exercise 6. You must prove (i) if $\angle D = \angle C$, then $AD = BC$ and (ii) if $AD = BC$, then $\angle D = \angle C$. You can simply sketch solutions for some of these exercises by writing a note that reminds you what to do. As an illustration, you could write "use SAS on the two triangles formed by the diagonals" for exercise 3. (Note that you should not use the Pythagorean Theorem here.) Exercise 11 is somewhat silly and follows immediately from Theorem 1.10. The last three exercises may require a bit more thought and are good practice for learning how to solve more difficult problems. Exercise 12 is not as scary as it might first appear; use Theorem 1.10 again. After using the hint, the exercise is completed referring to similar reasoning and SSS congruence. Exercise 13 can be solved by drawing in two extra lines, each of which gives the distance between the tracks. Finally, a solution to exercise 14 is somewhat similar to the proof of Theorem 1.10. Once again, a picture should indicate the required congruence relations. Also, as with exercise 10, two proofs are required for the solution to this problem.
3. It is a good idea to write some solutions more formally since this will be expected on the midterm. I will make note of this extra requirement for certain exercises and look for these solutions in your journal. For this section, write formal solutions for exercises 5, 8, and 14.
4. Here is a solution for exercise 9.

1D.9: In quadrilateral $ABCD$, if $\angle A = \angle C$ and $\angle B = \angle D$, then $ABCD$ is a parallelogram.

Proof: The figure below shows quadrilateral $ABCD$ with $\angle A = \angle C$ and $\angle B = \angle D$.

Since the sum of the interior angles of a quadrilateral is $360°$, we find that

$$360° = \angle A + \angle B + \angle C + \angle D = 2\big(\angle A + \angle D\big).$$

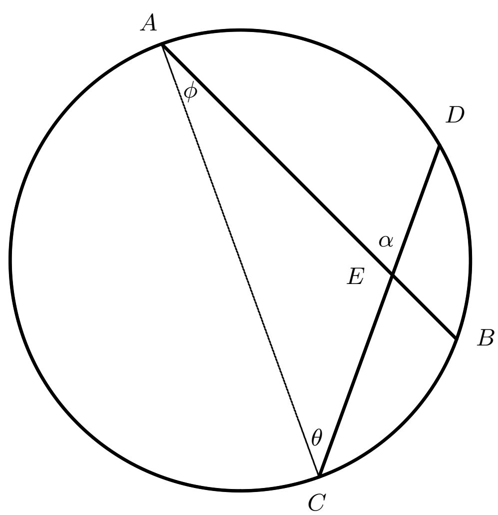
This shows that $\angle A$ and $\angle D$ are supplementary (note the typographical error in the text; we will find several of these over the next few weeks), and it follows that $AB \parallel CD$. Referring to exercise 8, we find that quadrilateral $ABCD$ is a parallelogram. (As we have done here, you may use the results of *previous* exercises in your solutions.) ■

Assignment 3: Read Section 1E in Isaacs.

1. Problem 1.14 may seem intimidating, but if you take your time the details are not too bad.
2. Do all the exercises at the end of the section. Exercise 1 and 4 follow by looking at areas and spotting figures with equal areas. There are several options for the requested formula in exercise 3; one involves sines (preferred) and another cotangents. Exercise 2 as stated (at least in my printing of the text) is incorrect. Triangle $\triangle ABC$ must be isosceles and BC must be its base. With this correction, area considerations lead to a solution rather quickly.
3. Write a formal solution to exercise 3 as well as a proof of the converse of Theorem 1.12.

Assignment 4: Read Section 1F through Problem 1.20 in Isaacs.

1. Do exercises 1, 2, 3, 4, and 9 at the end of the section. For exercises 3 and 4, you will need to use the familiar fact that a tangent line to a circle and a radius are perpendicular at the point of tangency.
2. Problem 1.20 is a standard max/min problem in calculus so take the time to solve the problem using calculus. Which solution method do you find to be easier?
3. Write a formal solution to exercise 3.
4. There are times when Isaacs uses one figure to do many things and this can lead to confusion. Figure 1.27 is one such example. A proof of Corollary 1.19 using a less cluttered figure appears below.



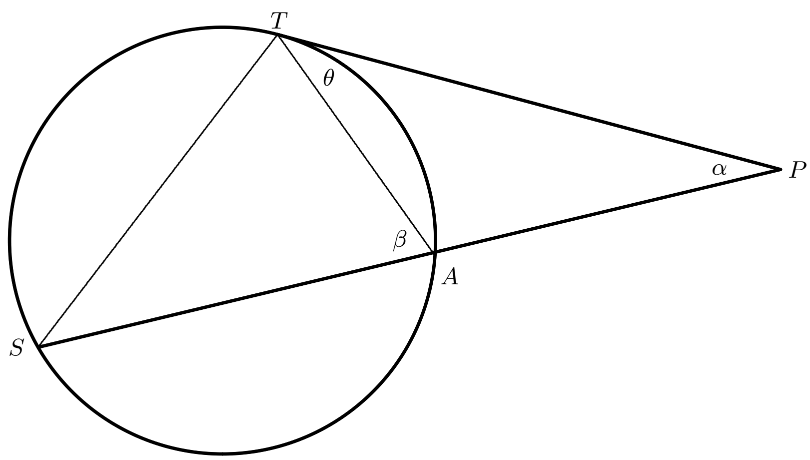
We want to find the acute angle α made by the chords AB and CD in the circle. Since α is an exterior angle of $\triangle ACE$, we find that

$$\alpha = \theta + \phi = \frac{1}{2}\,\text{arc}\,AD + \frac{1}{2}\,\text{arc}\,BC = \frac{1}{2}\big(\text{arc}\,AD + \text{arc}\,BC\big),$$

as desired. (You should draw a corresponding uncluttered figure for Corollary 1.18.)

Assignment 5: Read the rest of Section 1F in Isaacs.

1. Do exercises 5, 6, 7, and 12 at the end of the section.
2. Write a formal solution to exercise 6.
3. If you want to take on a challenge, give exercise 14 a shot. The final solution is fairly simple; the difficulty lies in determining what needs to be done.
4. Use the figure below to write a proof of Corollary 1.24 (which is exercise 13). The proof is little more than an equation beginning with the equality $\alpha = \beta - \theta$.



Assignment 6: Read Section 1G and Section 1H through the proof of Theorem 1.28.

1. You may skim Section 1G since it outlines some areas of higher mathematics we will not need but do pay attention to some of the ideas. However, do try to find a polynomial $P(x)$ with integer coefficients for which the number $x = \sqrt{2} + \sqrt{3}$ is a root. Can you do the same for $x = \sqrt{2} + \sqrt[3]{3}$?
2. Do exercises 1 and 3 at the end of the Section 1G. For exercise 1, you are asked to find an appropriate constant k so that $h = k \cdot RS \cdot ST$.
3. Write a formal solution to exercise 3 in Section 1G.
4. Denote the apothem (the usual spelling) of a regular n-gon by a. Show that the area A and perimeter P of the regular n-gon are given by $A = n\tan(\pi/n)a^2$ and $P = 2n\tan(\pi/n)a$. Note that $dA/da = P$ and compare this result with the corresponding circle formulas.
5. Do give the proofs of Lemma 1.29 and Theorem 1.28 careful thought.

Assignment 7: Finish reading Section 1H but omit Problem 1.34 and its solution.

1. Do exercises 1, 3, 4, 5, 6, 8, 9, and 11 in Section 1H. Note that exercise 8 is essentially Problem 1.30.
2. Write formal solutions for exercises 3 (do not forget the case in which there is a reflex angle) and 9.
3. Here is a proof for exercise 7. Applying Lemma 1.29 to $\triangle CPB$ and $\triangle CAQ$, respectively, we obtain
$$\frac{CX}{CP} = \frac{CQ}{CB} \quad \text{and} \quad \frac{CP}{CA} = \frac{CY}{CQ}.$$
Multiplying these equalities yields
$$\frac{CX}{CP} \cdot \frac{CP}{CA} = \frac{CQ}{CB} \cdot \frac{CY}{CQ} \quad \text{or} \quad \frac{CX}{CA} = \frac{CY}{CB}.$$
Using Lemma 1.29 again (the other direction of the if and only if statement), we find that $XY \parallel AB$.

Assignment 8: Read Section 2A, skimming the last few paragraphs if necessary.

1. Do exercises 3, 4, and 5 at the end of the section.
2. Write a formal solution for exercise 4. For the record, there are ways to solve this exercise that do not involve the hint. You might start by sketching a larger version of Figure 2.6.

Assignment 9: Read Section 2B and read Section 2C through the proof of Theorem 2.10.

1. You may skip the last three paragraphs of Section 2B unless you are intrigued by this approach.
2. Do exercises 1 and 3 in Section B and exercise 1 in Section C.
3. As an addition to the hint provided for exercise 2B.3, consider a line through the centroid of a triangle and parallel to one of the sides. This line divides the original triangle into a small triangle and a quadrilateral. What is the ratio of the area of the small triangle to the original triangle?
4. If you enjoyed the paragraphs on center of mass, you should try problem 2B.5. For the record, you can use analytic geometry to solve this exercise without thinking about centers of mass. Put the vertices of the triangle at $(0, 0)$, $(1, 0)$, and (a, b), with $a > 0$, and the point P at $(p, 0)$, where $0 < p < 1$. Find the coordinates of the point X then determine the desired ratio.
5. To fully engage with Theorem 2.10, draw an appropriate figure in which the angle at vertex A is obtuse.

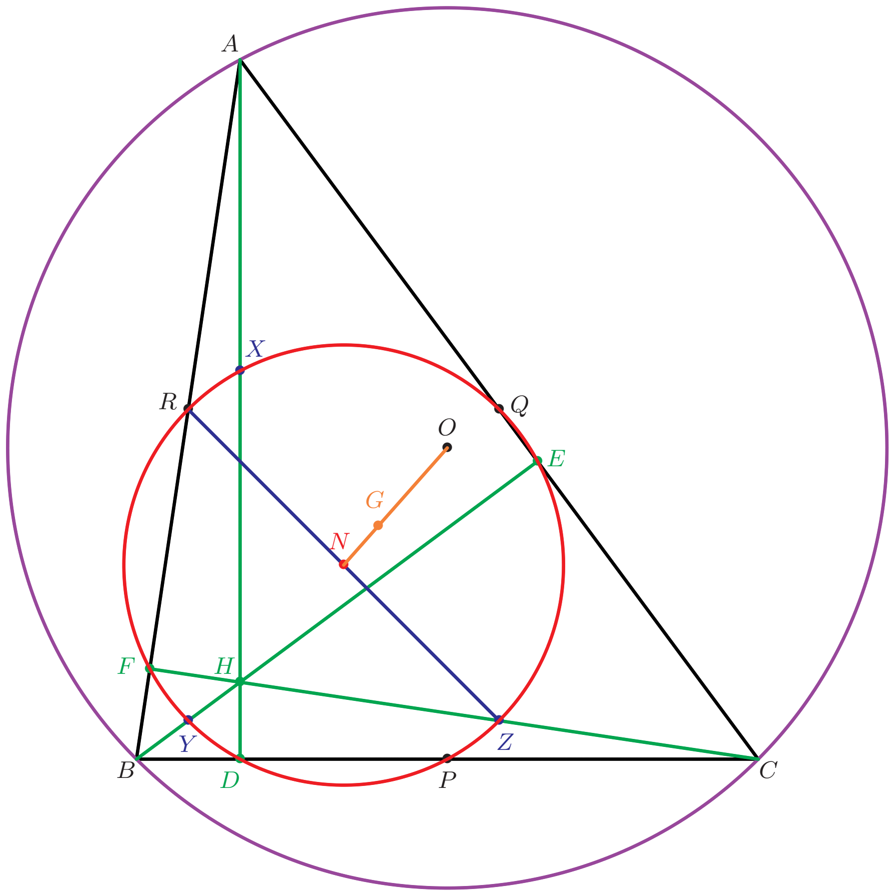
Assignment 10: Read the rest of Section 2C; this may take a while.

1. Spend some time carefully working through the proof of Theorem 2.12. After some thought, you will find that the proof is easier than it first appears.
2. The author briefly introduces transformational geometry in his solution to Problem 2.13. This is an important area of geometry but one that we will not be studying in this course. We can give proofs for Theorem 2.14 and Corollary 2.15 that are independent of this method. For Theorem 2.14, refer to Figure 2.12 and fill in the details behind the following equation, where R^* is the circumradius of triangle ABC (to distinguish this value from a point in the figure) and ρ is the radius of the nine point circle:

   $$2\rho = \frac{RQ}{\sin \angle RPQ} = \frac{\frac{1}{2}\,BC}{\sin \angle BAC} = R^*.$$

   To prove that $\triangle ABC$ and $\triangle HBC$ (Corollary 2.15) have the same circumradius, refer to Figure 2.11 and show that angles $\angle BAC$ and $\angle BHC$ are supplementary. The result then follows from the extended law of sines using the common side BC.
3. Solve exercise 2, then use this result, along with Theorem 2.10, to prove that N is the midpoint of segment OH. For the second part, look carefully at the Euler line and identify the location of the various points. Note that this approach avoids any use of transformational geometry.
4. Write a formal solution to exercise 5.

5. In case it is helpful, here is a large and colorful version of a nine point circle.



We are given triangle $\triangle ABC$ (in black) and its circumcircle (in purple).

Points P, Q, and R are the midpoints of the respective sides.

Point O is the circumcenter, point G is the centroid, and point H is the orthocenter.

Points D, E, and F are the feet of the altitudes (shown in green).

Points X, Y, and Z are the Euler points, halfway from a vertex to the orthocenter.

Point N is the center of the amazing nine point circle.

The line containing O, G, and N also passes through H; it is called the Euler line (in orange).

This circle goes through the three midpoints, the three altitude feet, and the three Euler points.

The center N of the nine point circle is the midpoint of $\overline{OH}$.

Assignment 11: Read Section 2D.

1. Do not let the somewhat messy algebra in the solutions to Problems 2.21 and 2.22 intimidate you.
2. Do all three exercises at the end of the section.
3. Write a formal solution for exercise 2. By the way, what would the circle look like if d were to have a value such as 2.999?
4. The length of the diagonal found in exercise 3 is the golden mean, usually denoted by ϕ. Look up this concept (in a textbook or on the Internet) to learn a little about it. Then use your pentagon to express $\sin 54°$ and $\cos 36°$ in terms of ϕ.

Assignment 12: Read Section 2E through the middle of page 77 (stop before Problem 2.30).

1. Note that Lemma 2.24 was exercise 1D.14.
2. Do exercises 1 and 2 at the end of the section.
3. In order to practice some basic trigonometry, give a direct proof that $\arccos(1/5) = 2\arctan\left(\sqrt{2/3}\,\right)$.

Assignment 13: Read the rest of Section 2E.

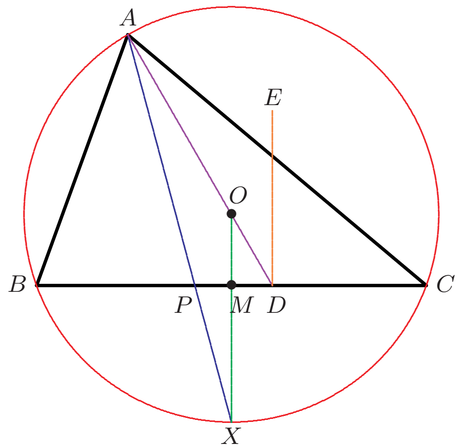
1. See if you can find a proof of Lemma 2.32 that does not involve $\angle B$ and $\angle P$.
2. Do exercises 3, 4, and 5 at the end of the section.
3. Write a formal solution to exercise 3.
4. The hint for exercise 5 refers to Lemma 2.32. You first need to draw a careful picture to see what needs to be done.

Assignment 14: Read Section 4A through the proof of Theorem 4.2.

1. In addition to reading this portion of Section 4A, also carefully read the statements for Theorem 2.36 and Theorem 2.46. (There is no need to read the proofs unless you are interested.) These two results present some interesting properties of triangles.
2. Suppose that $\triangle ABC$ is acute-angled so that all of its altitudes are Cevians. Use Ceva's Theorem to prove that the three altitudes are concurrent.
3. Do exercises 1, 2, and the first computation in 4 at the end of Section 4A. For exercise 1, the cases $T = I$ and $T = H$ are not too hard. For the case $T = G$, one option is to prove that $I = O$ and use Problem 2.30. Since it seems to be more complicated than the others, we present a detailed proof for the case in which $T = O$.

If the Gergonne point T and the circumcenter O in $\triangle ABC$ coincide, then the triangle is equilateral.

Proof: Since T is an interior point of $\triangle ABC$ and $T = O$, we know that $\triangle ABC$ is an acute-angled triangle (see the figure). We will prove that $AB = AC$; a proof that $BA = BC$ is similar.



Assume that $AB \neq AC$ and, without loss of generality, assume that $AB < AC$. Referring to the figure, the circumcircle of $\triangle ABC$ is shown in red, point M is the midpoint of segment BC, segment AX (with X on the circumcircle) is the bisector of $\angle A$, and OX is the perpendicular bisector of segment BC (see Problem 2.2). Since $AB < AC$, we find that $BP < PC$ (see Theorem 1.12) and $\angle C < \angle B$. It follows that P is between B and M and that

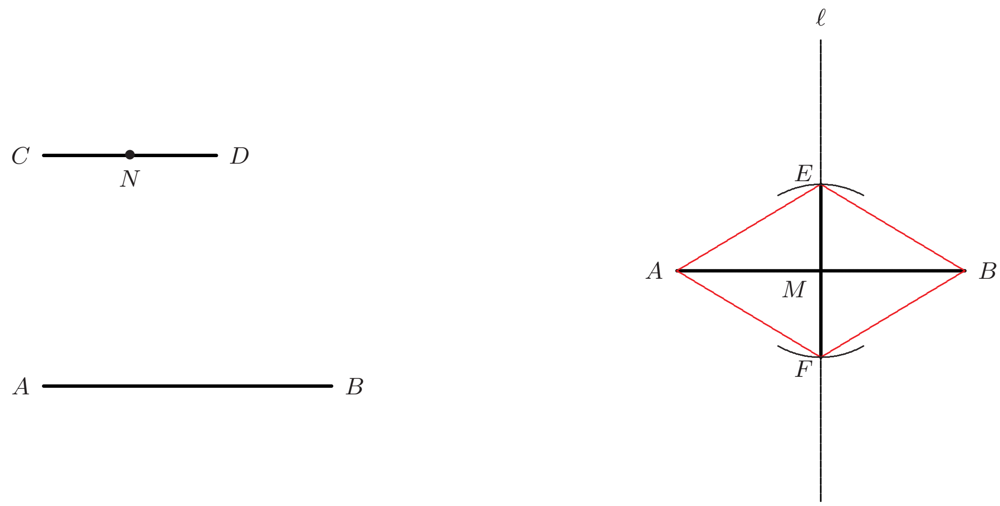
$$\angle APB = \pi - \angle B - \frac{\angle A}{2} < \pi - \angle C - \frac{\angle A}{2} = \angle APC.$$

From this last inequality, we conclude that $\angle APC$ is obtuse. Let D be the point where line $AO = AT$ meets segment BC and erect DE perpendicular to BC. We know that the incenter I lies on segment AP and, by the definition of the Gergonne point, that I lies on line DE. Since $\angle APD + \angle PDE > \pi$, the lines DE and AP meet below the line BC, a contradiction to the fact that I lies above line BC. It follows that $AB = AC$. As mentioned above, we can prove that $BA = BC$ with similar reasoning. We conclude that $\triangle ABC$ is equilateral. ∎

Assignment 15: Read Section 6A.

1. If it helps, you can think of the rules for constructions with compass and straightedge as being similar to rules for a sporting event such as basketball or a board game such as chess.
2. Do exercises 1, 2, 3, and 5 at the end of the section. There is a typographical error in exercise 5; it should be to construct $\triangle PAB$.
3. Write a formal solution to exercise 5.
4. For the record, what steps are required to construct the center of the nine point circle?
5. Ideally, you should obtain a compass and straightedge and carry out some of these constructions. If you choose not to do so, at least envision the steps that are needed. Once a construction has been shown to be possible, you can reference it when doing other constructions. However, it is important to remember that several (perhaps many) steps involving a compass and/or straightedge lie behind such references.

As an illustration of this idea, here is a solution for exercise 4: given two line segments, construct a rhombus whose diagonals have lengths equal to the lengths of the two given segments.



Given line segments AB and CD, we make the following constructions.

1. Construct the perpendicular bisector ℓ of segment AB. This also gives the midpoint M of AB. (This construction requires 3 steps; see Problem 6.3.)
2. Construct the midpoint N of CD. (This construction also requires 3 steps.)
3. Construct the circle with center M and radius CN, marking the points E and F where this circle meets line ℓ. (This construction requires 1 step *assuming* that our compass has a memory.)
4. Construct line segments AE, EB, BF, and FA. (This construction requires 4 steps.)

It follows easily (see exercise 1D.4 and Theorem 1.9) that $AEBF$ is a rhombus. Note that 11 steps are required for this construction, assuming that our compass has memory. In general, a construction that involves fewer steps is to be preferred over one that requires more steps.

Assignment 16: Read Section 6B, omitting Problem 6.9.

1. Read the solution to Problem 6.8 carefully and be certain you understand how the construction works. You should fill in the algebraic details that show that the construction does what it is intended to do.
2. Do exercises 1, 2, 3, and 4 at the end of the section. Note that exercise 3 is really just a special case of exercise 2, although it takes a little thought to recognize this.
3. Write a formal solution to exercise 4.

Assignment 17: Read Problems 6.10, 6.11, 6.12, 6.16, and 6.17 in Section 6C.

1. You do not need to study the other construction problems in this section unless you so desire.
2. Do exercises 1 and 3 at the end of the section, writing a formal solution for exercise 3.

Assignment 18: Read Sections 6D and 6E.

1. The main goal here is to give you a sense for some aspects of higher mathematics. Keep your eyes on the forest and do not get lost in the trees.
2. Do exercises 1 and 2 at the end of Section D. These exercises are intended to go quickly.
3. Many people fail to understand the meaning of "impossible" in the realm of mathematics. This seems to be especially true for the trisection of angles. As you have read, there are angles that cannot be trisected using only a compass and straightedge, but there are some angles that can be. Furthermore, there are angles that cannot be constructed using a compass and straightedge but which can be trisected using a compass and straightedge if the angle somehow magically appears. (See if you can show how to trisect the angle $3\pi/7$.) There are a number of trisection techniques using only a compass and straightedge that give very good approximations, and there are ways to trisect an angle using other tools. The existence of these methods, coupled with confusion and misunderstanding surrounding the mathematical meanings of "exact" and "impossible" and proper uses of the given tools, lead many people to continue to search for a way to solve this problem. You can refer to the book *The trisectors* by Underwood Dudley (Mathematical Association of America, 1994) to read about many of these ill-fated attempts. An article in a British journal (H. S. Carslaw, On the constructions which are possible by Euclid's methods, *Math. Gaz.* **83** (1910) 170-178) indicates that the frustration with people refusing to believe that the trisection problem is impossible is not a recent phenomenon.

Summary: The following list highlights the key results that you have studied in the Isaacs text. You should be able to explain these results in detail and be able to prove them. For the record, it is not expected that you can quickly write all of these proofs from memory. The idea is that you should be familiar enough with the ideas and techniques that you can (in a reasonable amount of time) figure out what to do. In some cases, you might want to record a phrase or two to help you remember how the proof goes.

1. Be familiar with ways of proving that triangles are congruent, including the HA criterion.
2. Know the properties of isosceles triangles.
3. Know the relationships between angles when two parallel lines are cut by a transversal.
4. The sum of the angles of a triangle is $180°$.
5. Opposite sides and opposite angles of a parallelogram are equal.
6. If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
7. If a quadrilateral contains a pair of opposite sides that are parallel and equal, then the quadrilateral is a parallelogram.
8. A quadrilateral is a parallelogram if and only if its diagonals bisect each other.
9. A parallelogram is a rectangle if and only if its diagonals are equal.
10. A parallelogram is a rhombus if and only if its diagonals are perpendicular.
11. A trapezoid is isosceles if and only if its base angles are equal.
12. Explain how an angle bisector in a triangle splits the opposite side. Is the converse true? (This is where a note reminding you to use area in the proof might be useful.)
13. Prove that three noncollinear points determine a unique circle.
14. Determine the measure (in terms of arcs) of an angle inscribed in a circle.
15. Opposite angles in inscribed quadrilateral are supplementary.

16. Determine the measure (in terms of arcs) of the angle made by two secant lines to a circle (or a secant line and a tangent line) meeting at an exterior point of the circle.
17. Determine the measure (in terms of arcs) of the angle made by two chords of a circle (or of a chord and a tangent line) meeting at an interior point of the circle.
18. Prove the Pythagorean Theorem.
19. Prove that corresponding sides of similar triangles are proportional.
20. The line joining the midpoints of two sides of a triangle is parallel to the base and half its size.
21. Prove the SSS and SAS similarity criteria.
22. Give a proof of the Pythagorean Theorem that involves similar triangles.
23. State and prove a result related to the products of certain lengths involving chords and secants for a given circle.
24. Explain why triangles have a circumcenter.
25. Prove the extended law of sines.
26. Discuss and verify the formula $4KR = abc$.
27. Explain why triangles have a centroid and show where it is located.
28. Explain why triangles have an orthocenter and show where it is located.
29. Discuss the nine-point circle. Include the location of the center and the length of its radius.
30. Prove the law of cosines.
31. Prove Heron's formula.
32. State Ptolemy's Theorem and use it to find the lengths of the diagonals in a regular pentagon.
33. Explain why triangles have an incenter.
34. Discuss a formula for the length of the radius of the incenter.
35. Show how to find the distance from the vertex of a triangle to the points of tangency the sides make with the incircle.
36. State and prove the law of tangents.
37. Show that if any two of the four key points (circumcenter, incenter, centroid, orthocenter) in a triangle coincide, then the triangle is equilateral.
38. State and prove Ceva's Theorem and give some illustrations of it.
39. Explain how to construct a line through a given point and parallel to a given line.
40. Explain how to construct the perpendicular bisector of a line segment.
41. Explain how to construct an angle bisector.
42. Explain how to locate the center of a given circle.
43. Explain how to construct a square with a given side length.
44. Explain the construction that divides a line segment into n equal parts.
45. Explain how to construct a tangent line to a circle given a point either on or outside of the circle.
46. Show how to construct the mean proportional of two given lengths.
47. Explain how to square a triangle and how to square a rectangle.
48. Describe the three unsolvable problems from Greek geometry.

Some practice problems for the midterm can be found on the website; solutions to these should go in your journal. The midterm will involve some problems from the Isaacs list, a few problems from the practice problems, and one or two novel problems.

Assignment 19: Read the preface and Chapter 1, as well as pp. 46–51, 109–112, 153–158, 190–194, and 202–203 in Heath.

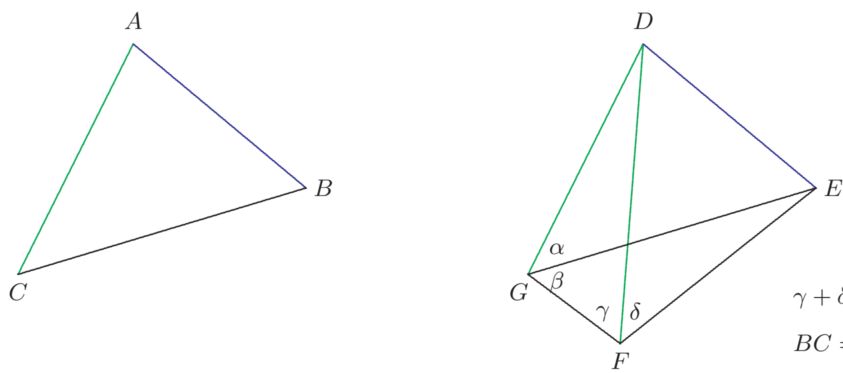
1. The preface should give you a sense of Heath's purpose for putting together this work and Chapter 1 provides you with the little we know about Euclid.
2. Pages 46–51 discuss the texts that have existed over the centuries and how we can try to determine what Euclid actually wrote. Remember that for close to two thousand years, books had to be recopied by hand using materials far different than those we have today. It is hard to imagine the effort involved in copying a single lengthy book.
3. Pages 109–112 discuss English translations of Euclid's *Elements* and how they influenced thought at the time.
4. Pages 153–158 give Euclid's definitions, postulates, and common notions, along with commentary on Definition 1 (point). You will find that the commentary is rather philosophical and this continues through all of the definitions. By reading the thoughts on Definition 1 you will get a sense for how people have thought about these ideas over the years. It is important to realize that the concept of "point" is not a simple one.
5. Pages 190–194 give commentary on Definition 23 (parallel lines) while pages 202–203 begin a lengthy discussion on the parallel postulate. We will have much more to say on this topic later in the course.
6. If time allows, you might find it interesting to skim other pages to get a sense of the history of this subject as well as an appreciation for the incredible amount of work that Heath devoted to this textbook.

Assignment 20: Read the statements and proofs of Propositions I.1 to I.15 in Heath. It takes a while to become accustomed to Euclid's style but it is fascinating to read something written well over two thousand years ago. You should seek the main ideas in each proof and think about ways to outline these in your own words and symbols.

1. Read the translation notes for I.1 to get a sense for the extreme care Heath has used.
2. Read the comments on I.2, skimming the ideas related to the figure.
3. Read the translation notes for lines 1–3 and line 44 as well as the first paragraph of notes for I.4.
4. Read the translation note for line 48 and the comments on Pappus' proof for I.5. The Dodgson quote has Euclid saying "Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?"
5. Read the translation notes for lines 1–6; personality differences do arise in this sort of work.
6. Read Philo's proof of I.8 and note its similarity with the proof on page 7 of Isaacs.
7. Read Apollonius' proof of I.10.

Assignment 21: Read the statements and proofs of Propositions I.16 to I.32 in Heath. Once again, look for the main ideas in each proof and think about ways to outline these in your own words and symbols.

1. Read the notes on I.16. Think about "lines" as arcs of circles on a sphere, where the centers of the circles are the same as the center of the sphere to visualize how the proof of I.16 could break down.
2. Read the notes on I.17, paying particular attention to the fact that there is only one perpendicular from a point to a line. Convince yourself that I.17 is the converse of the parallel postulate.
3. Read the first paragraph of the notes on I.18 to see one way to remember the order of I.18 and I.19.
4. Read the first section of notes on I.19 concerning logical deductions and how it applies here.
5. Read the note about donkeys knowing the truth of I.20 and the first of the alternative proofs of I.20.
6. Here is the sort of picture (for I.24) that helps me shorten and remember Euclid's proofs.



$\gamma + \delta > \gamma = \alpha + \beta > \beta$

$BC = EG > EF$

   Also, read the alternative proof of I.24 in the notes.

7. Read about the ambiguous case in the notes for I.26. You should recognize HA congruence in the last statement.
8. Read the first two paragraphs (Euclid is entering a new section) and the last paragraph (I.27 is the contrapositive of I.16) of notes after I.27.
9. Read all of the notes after I.29; these involve Playfair's Axiom.
10. Read the notes after I.31.
11. Read the Pythagorean proof section of notes after I.32.

Assignment 22: Read the statements and proofs of Propositions I.33 to I.48 in Heath.

1. Read the last part of the notes on I.34 concerning dividing a line into n equal parts. Use this method to divide a line into 5 equal parts, then compare your solution with the method used in Problem 6.8 in Isaacs. Do you prefer one method over the other?
2. Read the notes on I.34 concerning equality in a new sense.
3. Read the notes on I.37. Do you think you could fool children concerning area and perimeter? (Think about depth of water in wide versus skinny glasses.)
4. Read the result stated in the notes for I.38. Rather than working through the details of the proof, use the $\frac{1}{2}\, bc \sin A$ area formula to prove this result.
5. Read the first paragraph of the notes on I.44.
6. Read the last paragraph of the notes on I.45.
7. Read the notes for I.46. This result is implicitly used in the proof of I.48.
8. As you can see, there are many notes for I.47. You can read as many of these as you desire if you find them interesting. However, it is sufficient for you to read alternative proof I and Pappus' extension. It takes a while to understand the figures, but the details are not difficult once you have done so.

Assignment 23: Look back over Book I of Euclid.

1. Write out brief versions of each of the propositions and a sketch of the corresponding proof. Part of your oral final exam will be to state all 48 propositions and be able to give any proofs that are requested. Here is a portion of what my outline looks like but you may find a different approach works better for your learning style.

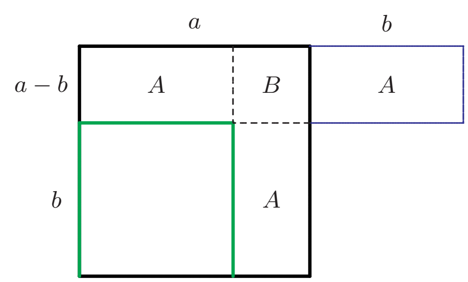
   I.9: bisect an angle: form equal sides, connect them, make equilateral $\triangle$ below the base, then use SSS
   I.10: bisect a segment: form equilateral triangle on the segment, then bisect vertex angle
   I.11: erect perpendicular: form equilateral triangle with the point as midpoint of the base, SSS
   I.12: drop perpendicular: circle through point on other side of line, bisect generated chord, SSS

Assignment 24: Continue reviewing and learning Book I of Euclid.

1. Fine tune your notes on Book I of Euclid and begin to store them in your memory. If you have not read all of the requested notes on various propositions, then you should fill in these gaps as well.

Assignment 25: Read the introduction to Book II of Euclid and the statements and proofs of Propositions II.1 to II.8 in Heath.

1. As Heath points out, each of these propositions can be represented by simple algebraic equations. For example, in Proposition II.3, we can let the whole straight line be $a + b$ and one of the segments be b. We then have $(a + b)b = ab + b^2$. Or we can let the whole straight line be a and the segment be x and obtain the equation $ax = x(a - x) + x^2$. However, the Greeks thought more in terms of geometry than in terms of algebra. One of the main reasons for this lies in the fact that the Greeks felt that geometry had a firm logical foundation whereas algebra did not since a solid theory of numbers (primarily irrational numbers) did not exist. A secure foundation for the real number system did not appear until the nineteenth century. Given our focus on algebra these days (and with perhaps little knowledge of philosophy), it may be difficult to appreciate this perspective.
2. Read the notes on II.2. Do you have any thoughts on the pedagogical issues raised here?
3. Read the notes on II.4. In reference to the question in the previous item, would an area interpretation for $(a+b)^2 = a^2 + 2ab + b^2$ be useful for students today? A related figure for $a^2 - b^2 = (a+b)(a-b)$ is given below.



$$a^2 - b^2 = A + B + A = (a + b)(a - b)$$

5. As you read more of these propositions, you might find yourself getting a little impatient with all of the details. It might be helpful to look at the figures and mentally shade in the areas that are supposed to be equal. With some practice, it becomes easier to see what is going on.
6. Read the note after II.5 about the geometric solution to quadratic equations. After reading the solution for the equation $ax - x^2 = b^2$, write out the minimal work required to construct (with compass and straightedge) the length x given the lengths a and b. You might also find the formulas for Pythagorean triples (given at the end of the note) interesting.
7. Read the portion of the notes on II.6 about the geometric solution to the quadratic equation $ax+x^2 = b^2$ and outline the minimal steps required to construct x in this case.
8. Before reading the proof of II.8, label the figure appropriately and see if you can look at the areas to convince yourself that $4(a + b)a + b^2 = (a + b + a)^2$. You should then be able to skim the proof to see how Euclid chose to word it.

Assignment 26: Read the statements and proofs of Propositions II.9 to II.14 in Heath.

1. The proofs of Proposition II.9 and II.10 look long and scary but there is actually very little going on. If you chase angles around in the figure, you should find many 45° angles and thus many isosceles triangles. The proofs follow quickly from these observations.
2. The corresponding algebraic equation for II.9 is

$$x^2 + (u-x)^2 = 2\Big[\Big(\frac{u}{2}\Big)^2 + \Big(\frac{u}{2} - x\Big)^2\Big],$$

   where $0 < x < u/2$. Replacing u with $a+b$ and x with b yields

$$a^2 + b^2 = 2\Big[\Big(\frac{a+b}{2}\Big)^2 + \Big(\frac{a-b}{2}\Big)^2\Big],$$

   where $0 < b < a$. Replacing a with $\alpha + \beta$ and b with $\alpha - \beta$ (assuming that $\alpha > \beta$) then gives

$$(\alpha+\beta)^2 + (\alpha-\beta)^2 = 2(\alpha^2 + \beta^2).$$

   However, keep in mind that the Greeks did not consider these equations as equivalent since each one corresponds to a different picture and breakdown into squares and rectangles.
3. Read the first two paragraphs of the notes following II.9 and think carefully about their implications.
4. Read the notes for II.10 that consider important deductions from II.9 and II.10. Compare the derivation for the length of a median discussed here with the derivation of the same formula using Stewart's Theorem on page 70 of Isaacs.
5. As with II.5, write out the minimal construction steps needed to find the point H in the figure for II.11.
6. Note that II.12 and II.13 are two versions of the law of cosines; be certain to see the connection between Euclid's wording and the modern equation. For these propositions, I find it much easier to label the sides of the triangles with single letters and then write equations that correspond to each step of the proof.
7. Read the last part of each of the notes for II.12 and II.13 as these short proofs illustrate a nice application of I.25.
8. Read the notes for II.14. Being able to construct a square whose area is equal to any given rectilineal area is quite a feat given the use of basic results only.

Assignment 27: Read the preface and Chapter 1 of Wolfe.

1. The preface presents some interesting thoughts on pedagogy and philosophy.
2. The implicit assumptions made in the proof of I.16 are very important. You might find it helpful to use a tennis ball as your surface and draw some triangles on it. (Remember that lines in this case are circles with the same center as the sphere.) You can then visualize how the options at the bottom of page 8 can occur. In the same way, you should be able to draw a triangle for which Pasch's Axiom fails to hold.
3. Do not get bogged down in the details of the proof on pages 11 and 12. It is important to realize that continuity is a more complicated concept than you might expect. Are matter and time continuous? Is the concept of continuity a result of human thought?
4. Read through the axioms of Hilbert. Starting geometry at this level is very abstract and does not have wide appeal; this is why Isaacs chose to start his textbook in a different place.

Assignment 28: Read Sections 10 through 23 in Chapter 2 of Wolfe.

1. Read the arguments that are presented carefully and think about the implications of these results. If you are interested, pages 204–213 in Heath present very similar material but with more detail.
2. Do the two exercises that appear in the reading. You should be able to do Exercise 1 by referring to results in Book I. Although the proofs for Exercise 2 are not difficult, do write the details carefully. Convince yourself that the statement in 2b is the contrapositive of the statement in 2a.
3. Referring to Figure 8 and the corresponding hypotheses, you should be able to prove that PO and RQ are both perpendicular CD. Write out the details then use the existence of a quadrilateral with four right angles to finish the proof in a slightly different way.

Assignment 29: Finish reading Chapter 2 of Wolfe.

1. Read Legendre's arguments carefully as they present some interesting ideas. I suggest that you read some of the material on pages 213–219 in Heath as he gives more information on Legendre's attempts to prove the parallel postulate.
2. Using ideas from I.1 to I.28, how would you perform the construction needed for Figure 14? In the same way, how would you construct the squares required in the paragraph after Figure 16?
3. Use the Fifth Postulate to prove that "through any point within an angle less than two-thirds of a right angle, there can always be drawn a straight line which meets both sides of the triangle."
4. Show that the italicized result stated at the end of Section 24 is the contrapositive of the previous italicized result. What assumption do you have to make in order for this relationship to be valid?

Assignment 30: Read Chapter 3 of Wolfe.

1. I hope that you find the historical development of non-Euclidean geometry rather interesting. As you read through this version of the story, remember that writing and publishing books was more difficult during this time period, that the time lag for communication was much different than it is today, and that the language barrier was more pronounced. Also, think about the human element of the story, the highs and lows of the people involved in discovering and communicating these results.

Assignment 31: Read Sections 36 through 39 in Chapter 4 of Wolfe.

1. As will be immediately evident, I have taken Wolfe's ideas and put them in a statement/reason proof style. This may or may not be beneficial for you but give it a try and study each step carefully. You can certainly read the text online or (better yet) purchase a copy of the textbook through Dover if you want to see Wolfe's arguments in paragraph form.
2. As you can see in Section 38, proofs in hyperbolic geometry require quite a bit of care and attention to detail. Be certain you understand each step in the proofs.
3. Do Exercise 1.

Assignment 32: Read Sections 40 and 41 in Chapter 4 of Wolfe.

1. Although it may seem "superfluous" to do so, try drawing some figures to indicate what is meant by the comment after the proof of Theorem 8.
2. Do problems 2–4 at the end of Section 40. You should find that these solutions are rather short. If you have time, you can start looking at the other problems.

Assignment 33: Go back over the material in Chapter 4 that you have been reading and fill in any gaps.

1. Prove that SASSS, SASAS, AASAS, and ASASA are congruence criteria for convex quadrilaterals. Draw some pictures to understand what these mean, then prove them by including the diagonals and using triangle congruence properties. We will use these facts in the next few sections and let CPCQE be an abbreviation for corresponding parts of congruent quadrilaterals are equal. The symbols $\square ABCD$ refer to the quadrilateral $ABCD$.
2. Do problems 5–7 at the end of Section 40.
3. Write formal solutions for problems 5 and 7. You can think of the line in problem 5 as the bisector of the angle at the ideal point of the figure and compare this result with Exercise 1D.14 in Isaacs. Note (as in really ponder this) that problem 7 is relevant to the discussion in Section 41.

Assignment 34: Read Sections 42 and 43 in Wolfe.

1. As you read the last part of the proof of Theorem 14, you might be reminded of the notes for I.19 on page 284 of Heath.
2. Do problems 8–12 at the end of Section 43. You should find that the few first problems can be solved quickly using previous results.
3. Write formal solutions for problems 11 and 12.

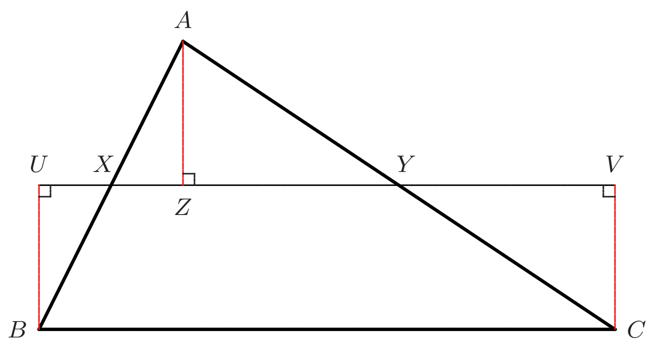
Assignment 35: Continue working on any problems that you have yet to solve and/or reviewing results in Chapter 4.

1. Here is a partial solution for problems 13 and 14 at the end of Section 43.

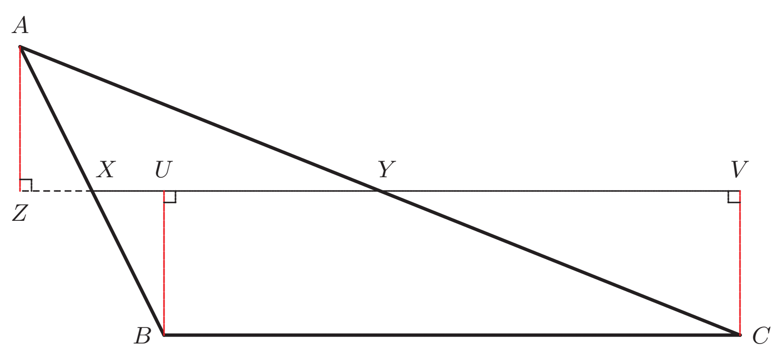
In the figure below, points X and Y are the midpoints of sides AB and AC, respectively, in acute-angled $\triangle ABC$. Perpendiculars are dropped from B and C to the line XY, meeting this line at points U and V, respectively. We have also dropped AZ perpendicular to XY. It is easy to see that $\triangle UXB \cong \triangle ZXA$ and $\triangle VYC \cong \triangle ZYA$ by AAS and we find that $BU = AZ = CV$. It follows that $BCVU$ is a Saccheri Quadrilateral with base UV and summit BC. The rest of problem 13 follows from Theorem 11. Since the summit is larger than the base (Corollary 16), we obtain

$$2 \cdot XY = 2 \cdot XZ + 2 \cdot ZY = UZ + ZV = UV < BC,$$

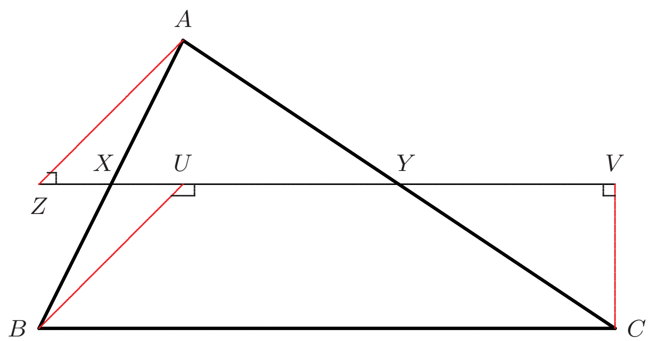
which is the desired result for problem 14.



Why is this just a partial solution? Well, there are other cases to consider. If $\angle A$ is right or obtuse, the above figure and proof work just fine. When $\angle B$ is obtuse, the figure could look like



We leave the proof in this case, as well as the case in which $\angle B$ is right, to the student. However, we still need to be careful and not let our Euclidean geometry influence our thinking. Consider the following figure for acute-angled triangles. We have changed the locations of the points where the dropped perpendiculars meet the line XY. The figure now seems odd but there is no reason that it cannot be correct in hyperbolic geometry. Does the above proof still work for this figure? After answering this question, you should spend some time thinking about the relative locations of the points. For instance, can the point V lie between X and Y when U does? Can the points U and Z be on the same side of X? To answer these questions, I suggest that you consider triangles and use Proposition I.16.



2. Do problems 15 and 16 at the end of Section 43. You may find problem 13 useful for problem 15 and then problem 15 useful for problem 16.
3. Write a formal solution for problem 16.

Assignment 36: Read Section 44 in Wolfe.

1. Do problems 17 and 18 at the end of Section 44.
2. Write a formal solution for problem 17.
3. Do some Internet searching to find the Poincaré disk model for hyperbolic geometry. Spend a few minutes pondering it and looking at some of the images corresponding to this model.

The oral final exam covers Book I of Euclid and essentially Sections 42–44 of Wolfe. You need to be able to state all 48 propositions of Book I, in abbreviated form, and be prepared to give the proofs of these propositions, again in abbreviated form. I will ask you to state several propositions, usually six or so at a time, then pause and have you outline the proof of one or two of them. You need to be able to state and prove these propositions efficiently as part of your grade will depend on how many questions you can answer during your oral exam. For instance, you can (and should) just say SSS congruency for Proposition 8 and you should be able to prove Proposition 16 in less than a minute by quickly drawing some lines, labeling a figure, and explaining what happens. As for the hyperbolic geometry material, you should know the properties (with proofs) of Saccheri and Lambert quadrilaterals as well as a proof that the sum of the angles of a triangle is less than two right angles.