Problem: For each positive integer n, the formula

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

is valid.

Proof: (formal style; it is good to do a few proofs this way) We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n such that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that $1 \in S$. Suppose that $k \in S$ for some positive integer k. We then have

$$\begin{aligned}
&1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) && \text{(substituting } k+1 \text{ for } n\text{)} \\
&\quad = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2) + (k+1)(k+3) && \text{(include extra term)} \\
&\quad = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) && \text{(since } k \in S\text{)} \\
&\quad = \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) && \text{(factoring)} \\
&\quad = \frac{k+1}{6}(k+2)(2k+9) && \text{(more factoring)} \\
&\quad = \frac{(k+1)(k+2)\big(2(k+1)+7\big)}{6}. && \text{(the form we want)}
\end{aligned}$$

This shows that $k+1 \in S$. By the Principle of Mathematical Induction, it follows that $S = Z^+$. Hence,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

for all positive integers n. ■

Proof: (informal style; more common in textbooks) The formula given in the statement of the problem is clearly true for $n = 1$. Suppose that the formula is valid for some positive integer k. Then

$$\begin{aligned}
1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\
&= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) \\
&= \frac{(k+1)(k+2)\big(2k+9\big)}{6},
\end{aligned}$$

showing that the formula is valid for $k+1$ as well. The result now follows by the Principle of Mathematical Induction. ■

Here are three PMI proofs of this same result, each with one or more errors; be certain you can spot the errors.

Proof: We will use the Principle of Mathematical Induction. Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that the formula works when $n = 1$. Suppose that $k \in S$ for some positive integer k. We then have

$$\begin{aligned} 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\ &= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) \\ &= \frac{(k+1)(k+2)(2k+9)}{6}, \end{aligned}$$

so $k+1 \in S$. By the Principle of Mathematical Induction,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

for all positive integers n. ■

Proof: We will use the Principle of Mathematical Induction. Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that the formula works when $n = 1$. Now suppose that the formula is valid for every positive integer k. Then

$$\begin{aligned} 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\ &= \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) \\ &= \frac{(k+1)(k+2)\big(2(k+1)+7\big)}{6}, \end{aligned}$$

so the formula works for all n. ■

Proof: We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n such that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Since $1 \cdot 3 = (1 \cdot 2 \cdot 9)/6$, it is clear that $1 \in S$. Suppose that $k \in S$ for some positive integer k. We then have

$$\begin{aligned} 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+3) &= \frac{(k+1)(k+2)\big(2(k+1)+7\big)}{6} \\ \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) &= \frac{(k+1)(k+2)(2k+9)}{6} \\ \frac{k+1}{6}\left(2k^2 + 7k + 6k + 18\right) &= \frac{k+1}{6}\,(2k^2 + 13k + 18). \end{aligned}$$

This shows that $k+1 \in S$. By the Principle of Mathematical Induction, it follows that $S \in Z^+$. Hence,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

for all positive integers n. ■

Here are three correct proofs for a different result; study them carefully.

Theorem: For each positive integer n, the integer $3^{2n+1}+2^{n+2}$ is divisible by 7.

Proof 1: We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n for which $3^{2n+1}+2^{n+2}$ is divisible by 7. When $n=1$, we see that $3^3+2^3=35$ is divisible by 7. This shows that $1\in S$. Now suppose that $k\in S$ for some positive integer k. Since $3^{2k+1}+2^{k+2}$ is divisible by 7, there exists an integer q such that $3^{2k+1}+2^{k+2}=7q$. We then have (using one of several options)

$$\begin{aligned}3^{2k+3}+2^{k+3}&=3^2\cdot 3^{2k+1}+2\cdot 2^{k+2}\\&=7\cdot 3^{2k+1}+2\big(3^{2k+1}+2^{k+2}\big)\\&=7\cdot 3^{2k+1}+2(7q)\\&=7\big(3^{2k+1}+2q\big),\end{aligned}$$

revealing that 7 divides $3^{2k+3}+2^{k+3}$. This means that $k+1\in S$. By the Principle of Mathematical Induction, $S=\mathbb{Z}^+$. Hence, the integer $3^{2n+1}+2^{n+2}$ is divisible by 7 for each positive integer n. ■

Proof 2: We will use the Principle of Mathematical Induction. For each positive integer n, let P_n be the statement that $3^{2n+1}+2^{n+2}$ is divisible by 7. Since $3^3+2^3=35$ is divisible by 7, it is clear that P_1 is true. Suppose that P_k is true for some positive integer k. Since $3^{2k+1}+2^{k+2}$ is divisible by 7, there exists an integer q such that $3^{2k+1}+2^{k+2}=7q$. We then have (using one of several options)

$$\begin{aligned}3^{2k+3}+2^{k+3}&=3^2\cdot 3^{2k+1}+2\cdot 2^{k+2}\\&=9\big(7q-2^{k+2}\big)+2\cdot 2^{k+2}\\&=63q-7\cdot 2^{k+2}\\&=7\big(9q-2^{k+2}\big),\end{aligned}$$

revealing that 7 divides $3^{2k+3}+2^{k+3}$. This means that P_{k+1} is true. By the Principle of Mathematical Induction, all of the P_n statements are true, that is, the integer $3^{2n+1}+2^{n+2}$ is divisible by 7 for each positive integer n. ■

Proof 3: The statement is easily seen to be true when $n=1$. Suppose that $3^{2k+1}+2^{k+2}$ is divisible by 7 for some positive integer k and choose an integer q such that $3^{2k+1}+2^{k+2}=7q$. We then have

$$\begin{aligned}3^{2k+3}+2^{k+3}&=3^2\cdot 3^{2k+1}+2\cdot 2^{k+2}\\&=9\big(3^{2k+1}+2^{k+2}\big)-7\cdot 2^{k+2}\\&=9\big(7q\big)-7\cdot 2^{k+2}\\&=7\big(9q-2^{k+2}\big),\end{aligned}$$

revealing that 7 divides $3^{2k+3}+2^{k+3}$. By the Principle of Mathematical Induction, the integer $3^{2n+1}+2^{n+2}$ is divisible by 7 for each positive integer n. ■