It is sometimes necessary to find a function given information about its derivative. An equation that involves the derivative of a function is known as a **differential equation**. A solution to a differential equation is a function that satisfies the equation. As an extremely simple example, suppose that we need to find a function f with the property that $f'(x) = 2x$. It is clear that the function $f(x) = x^2$ satisfies this differential equation. Note that $f(x) = x^2 + 1$ and $f(x) = x^2 - 8$ also satisfy the differential equation. Using the Mean Value Theorem, we can prove that every function that satisfies $f'(x) = 2x$ must be of the form $f(x) = x^2 + C$, where C is a constant. However, if further information about the function is known, a unique solution can be determined. Suppose that $f'(x) = 2x$ and $f(1) = 4$. The only function f that satisfies both of these conditions is $f(x) = x^2 + 3$.

Now consider the differential equation $f'(x) = 2f(x)$. This differential equation differs from the one considered earlier since the unknown function appears on both sides of the equation. A solution to this equation is a function whose derivative is the same as the original function multiplied by two. Thinking over the list of derivative formulas indicates that a solution is $f(x) = e^{2x}$. Where should the generic constant C go? A quick check shows that $f(x) = e^{2x} + C$ does not satisfy the equation $f'(x) = 2f(x)$, but $f(x) = Ce^{2x}$ does. In general, if both C and k are constants, the chain rule gives

$$\frac{d}{dx}Ce^{kx} = Ce^{kx} \cdot k = k\big(Ce^{kx}\big).$$

This shows that the derivative of the function Ce^{kx} is k times itself. This discussion (combined with the Mean Value Theorem again) essentially proves the following theorem.

Theorem: Let G be a differentiable function and let k be a constant. If $G'(x) = kG(x)$ for all x, then $G(x) = Ce^{kx}$, where C is a constant. It should be clear that $C = G(0)$. ■

Here are two examples that illustrate the type of differential equation considered in the theorem.

If $f'(x) = 4f(x)$ and $f(0) = 20$, then $f(x) = 20e^{4x}$.

If $s'(t) = -s(t)$ and $s(0) = 5$, then $s(t) = 5e^{-t}$.

It is important to remember that the type of differential equation considered in the theorem is different than differential equations in which the derivative of the function is explicitly given. If the derivative is explicitly given, then the solution can be found using antiderivative formulas or techniques of integration; you should have a fair amount of experience (especially from Calculus II) solving these types of "differential equations." In addition to antiderivative formulas, you must be familiar with integration by parts and integration using partial fractions. (Look these up if you do not remember them!)

Using a little bit of algebra, we can solve slightly more complicated differential equations.

Problem: Find a function A such that $A'(t) = 10 - 2A(t)$ and $A(0) = 40$.

Solution: The first step is to factor out the coefficient of $A(t)$, then recognize that $A(t)$ and $A(t) - 5$ have the same derivative (include this step in your homework problems):

$$A'(t) = -2\big(A(t) - 5\big) \;\Rightarrow\; \frac{d}{dt}\big(A(t) - 5\big) = -2\big(A(t) - 5\big).$$

The second equation is of the form given by the theorem with $G(t) = A(t) - 5$ and $k = -2$. Since $G(0) = A(0) - 5 = 35$, it follows that $A(t) - 5 = G(t) = 35e^{-2t}$. Thus the solution to the differential equation is $A(t) = 35e^{-2t} + 5$.

Exercises

1. Evaluate each of the following indefinite integrals. Recall (or learn) that the hyperbolic trigonometric functions are defined by $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. These functions have simple derivative formulas (find them), which you will need for part (c) of this problem and for later use in this course.

   a) $\displaystyle\int x\cos x\,dx$ **b)** $\displaystyle\int \ln x\,dx$ **c)** $\displaystyle\int x\sinh x\,dx$

   d) $\displaystyle\int \frac{3x-1}{x^2+4}\,dx$ **e)** $\displaystyle\int \frac{x}{x^2-2x-3}\,dx$ **f)** $\displaystyle\int \frac{12}{x^2+6x+5}\,dx$

2. Solve each of the following simple differential equations.

   a) $f'(x) = 3x^2,\ f(1) = 2$ **b)** $g'(x) = \dfrac{2}{x},\ g(1) = 5$ **c)** $h'(x) = \sqrt{x},\ h(4) = 6$

   d) $F'(x) = 3x^2 - 8x,\ F(0) = 1$ **e)** $G'(x) = xe^{-x},\ G(0) = 7$ **f)** $H'(x) = \dfrac{10}{(x+2)(3-x)},\ H(2) = 0$

   g) $f'(t) = e^{2t},\ f(0) = 3$ **h)** $g'(t) = 3\sin(2t),\ g(\pi) = 5$ **i)** $h'(t) = \dfrac{t}{\sqrt{t^2+1}},\ h(0) = 2$

3. Solve each of the following differential equations.

   a) $f'(t) = -2f(t),\ f(0) = 8$ **b)** $g'(t) = 4g(t),\ g(0) = 5$ **c)** $F'(x) = 4 - F(x),\ F(0) = 30$

   d) $A'(t) = 8 + 2A(t),\ A(0) = 20$ **e)** $B'(t) = 15 - 5B(t),\ B(0) = 2$ **f)** $S'(t) = 80 - \dfrac{1}{3}S(t),\ S(0) = 40$

4. A 300 gallon tank contains 200 gallons of brine (salt dissolved in water) with a concentration of 1/4 pound of salt per gallon of water. A brine containing 1 pound of salt per gallon of water runs into the tank at the rate of 4 gallons per minute, and the well-stirred mixture runs out of the tank at the same rate. When will there be 100 pounds of salt in the tank? *Hint*: Let $A(t)$ denote the number of pounds of salt in the tank at time t minutes. Then $A'(t)$ represents the rate (in lbs/min) at which the amount of salt in the tank changes. This rate can be determined by finding the rate in (this is easy) minus the rate out (this is not quite so easy). We thus find a differential equation for the function A; it is similar in form to part (f) of the previous problem. Solve the differential equation for A, then answer the question posed in the problem.