A Brief Summary of Math 126

The **integral** of a continuous function f on an interval $[a, b]$, denoted by $\int_a^b f(x)\,dx$, is defined by

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^n f\left(a + \frac{i}{n}(b - a)\right) \cdot \frac{b - a}{n}.$$

The definition of the Riemann integral (the usual calculus integral) is actually more involved than this but you are not expected to know the general definition (which allows for functions that are not continuous) for the written exam. Assuming that f is nonnegative on $[a, b]$, the number $\int_a^b f(x)\,dx$ represents the **area** under the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$. However (as you should recall), the integral has a number of more interesting interpretations and applications.

Evaluating integrals using the definition is more complicated than doing so for derivatives. The sum formulas

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

(which you should be familiar with) are useful when doing these computations. As a result of this difficulty, the discovery of a quick way to find the value of an integral is considered to be the beginning of what we call calculus. This result, usually given in two parts, is known as the **Fundamental Theorem of Calculus**.

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for all x in $[a, b]$.

If f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

The second version of the FTC is the one that makes evaluating many integrals quite easy; determine an antiderivative and plug in the endpoints. The first version often gives students more trouble since the definition of the function F using an integral is rather abstract; you should practice some of these. However, both versions essentially state that integration and differentiation are inverse processes.

The **Mean Value Theorem for Integrals** is often used to help prove the FTC.

If f is continuous on $[a, b]$, then there exists a point c in $[a, b]$ such that $f(c)(b - a) = \int_a^b f(x)\,dx$.

The value $f(c)$ is often called the average value of f on $[a, b]$. You should be able to give a geometric interpretation (involving area) of this result for continuous nonnegative functions.

You should be familiar with the following basic antiderivative formulas. (The $+C$ has been omitted.)

$$\int u^r\,du = \frac{1}{r+1}\,u^{r+1},\ r \neq -1; \qquad \int \frac{du}{u} = \ln|u|; \qquad \int e^u\,du = e^u;$$

$$\int \sin u\,du = -\cos u; \qquad \int \cos u\,du = \sin u; \qquad \int \sec^2 u\,du = \tan u;$$

$$\int \sec u \tan u\,du = \sec u; \qquad \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin(u/a); \qquad \int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan(u/a).$$

When seeking an antiderivative, you should first assume that some easy technique will work. These include basic formulas, algebra (multiply out, long division, completing the square, splitting up), and u-substitution. If none of these seem to work, then consider **integration by parts** $\left(\int u\,dv = uv - \int v\,du\right)$, **trigonometric substitution** (so you need to know some basic identities), or **partial fractions** (know the form for the parts and how to determine the coefficients. You should practice enough of these integration problems (without knowing which section of the book they came from) so you know how to approach them.

Integration can be used to solve problems involving **distance** $\left(\int_{t_0}^{t_1} |v(t)|\, dt\right)$, **area**, **volume**, and **arc length** $\left(\int_a^b \sqrt{1 + (f'(x))^2}\, dx\right)$. (There are other applications such as work, force of a liquid, and center of mass but not everyone covers all of these.) Remember that two methods (cross-sections and shells) can be used to find the volumes of solids of revolution but the method of cross-sections applies to objects like pyramids where the cross-sections are not composed of circles. Rather than memorize formulas for these, you should understand the basic ideas behind the formulas and apply them to any given situation.

You should also be familiar with **improper integrals** $\left(\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx\right)$ and review the basic ideas behind the **trapezoid rule** and **Simpson's rule** for numerical integration.

As for sequences and series, start by not freaking out and review the basic ideas behind each of these concepts. Know what a sequence is and how to compute the limit of a convergent sequence, including standard sequences such as $\left\{\sqrt[n]{n}\right\}$ and $\left\{\left(1 + \frac{a}{n}\right)^n\right\}$. Understand that an infinite series $\sum_{k=1}^\infty a_k$ is really two sequences, the sequence of terms $\{a_k\}$ and the sequence of **partial sums** $\{s_n\}$, where $s_n = \sum_{k=1}^n a_k$. The **Divergence Test** states that a series diverges if the sequence of terms $\{a_k\}$ does not converge to 0. However, it is possible for the series $\sum_{k=1}^\infty a_k$ to diverge even when $\{a_k\}$ does converge to 0. For this reason, we develop further tests such as **Integral Test**, **Comparison Test**, **Limit Comparison Test**, **Alternating Series Test**, **Ratio Test**, and **Root Test**. You need to be familiar with each of these tests and (as with techniques of integration) you need to be able to decide which is the appropriate test to use for a given series.

Two large classes of series are the **geometric series** $\sum_{k=0}^\infty ar^k$ (which converges to $a/(1-r)$ if $|r| < 1$ (that is, the first term over 1 minus the common ratio) and diverges if $|r| \geq 1$) and the **p-series** $\sum_{k=1}^\infty 1/k^p$ (which converges if $p > 1$ and diverges if $p \leq 1$).

The series $\sum_{k=1}^\infty a_k$ **converges absolutely** if $\sum_{k=1}^\infty |a_k|$ converges. (It then follows that $\sum_{k=1}^\infty a_k$ converges.)

The series $\sum_{k=1}^\infty a_k$ **converges conditionally** if $\sum_{k=1}^\infty a_k$ converges and $\sum_{k=1}^\infty |a_k|$ diverges.

To prove that a series is conditionally convergent, two proofs are needed; one proving that $\sum_{k=1}^\infty |a_k|$ diverges (often a comparison test) and another (probably the AST) proving $\sum_{k=1}^\infty a_k$ converges.

A **power series** is an expression of the form $\sum_{k=0}^\infty c_k(x - a)^k$; it is essentially an infinite degree polynomial and behaves pretty much like a polynomial. The c_k's are the coefficients and a is the center. The **interval of convergence** of a power series is the set of all values of x for which the series converges while the **radius of convergence** is the number ρ so that the series converges for $|x - a| < \rho$ and diverges for $|x - a| > \rho$. The radius of convergence is often found most easily using the Ratio Test. Note that endpoints must be checked to determine the interval of convergence.

Given an infinitely differentiable function f, the series $\sum_{k=0}^\infty \frac{f^{(k)}(a)}{k!}(x - a)^k$ is known as the **Taylor series** for f centered at a. If $a = 0$, the series is called a **Maclaurin series**. You should know the Maclaurin series

$$e^x = \sum_{k=0}^\infty \frac{1}{k!}\, x^k, \qquad \sin x = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}, \qquad \cos x = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}\, x^{2k}.$$

as well as how to find a Taylor series for a given function and center.