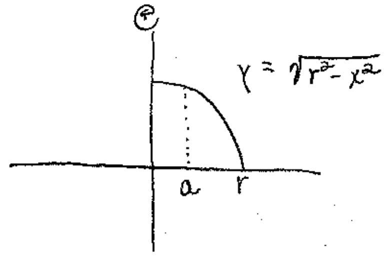
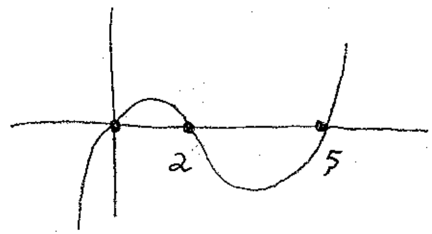
1. There are various options here; we will revolve the region under the curve $y = \sqrt{r^2 - x^2}$ on $[a, r]$ around the y-axis then double the result. Using the shell method,



$$V = 2\int_a^r 2\pi x\sqrt{r^2 - x^2}\,dx = -\frac{4\pi}{3}\left(r^2 - x^2\right)^{3/2}\Big|_a^r = \frac{4\pi}{3}\left(r^2 - a^2\right)^{3/2}$$

The volume of the remaining solid is $\frac{4\pi}{3}\left(r^2 - a^2\right)^{3/2}$ cubic units.

2. The region is sketched to the right. Since the curve is below the x-axis on $[2, 5]$, the total area is



$$A = \int_0^2 x(x-2)(x-5)\,dx - \int_2^5 x(x-2)(x-5)\,dx$$
$$= \int_0^2 (x^3 - 7x^2 + 10x)\,dx - \int_2^5 (x^3 - 7x^2 + 10x)\,dx$$
$$= \left(\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 + 5x^2\right)\Big|_0^2 - \left(\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 + 5x^2\right)\Big|_2^5$$
$$= 4 - \frac{56}{3} + 20 - \left(\frac{625}{4} - \frac{875}{3} + 125\right) + \left(4 - \frac{56}{3} + 20\right)$$
$$= -77 + \frac{763}{3} - \frac{625}{4}$$
$$= -77 + 254 + \frac{1}{3} - 156 - \frac{1}{4} = 21\tfrac{1}{12}$$

The total area of the region is $21\frac{1}{12}$ square units.

3. We first do long division

$$\begin{array}{r} x \\ x^2+4 \overline{\big)\, x^3+1} \\ \underline{x^3+4x} \\ -4x+1 \end{array}$$

It follows that

$$\int \frac{x^3+1}{x^2+4}\,dx = \int \left(x + \frac{-4x+1}{x^2+4}\right) dx$$

$$= \int \left(x - \frac{4x}{x^2+4} + \frac{1}{x^2+4}\right) dx$$

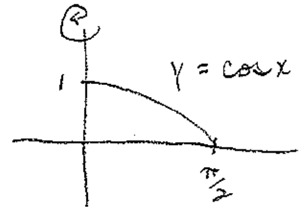
$$= \frac{1}{2}x^2 - 2\ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$$

4. If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

If f is continuous on $[a,b]$ and F is defined by $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for all x in $[a,b]$.

By the second version, the function f defined by $f(x) = \int_1^x \frac{2^t}{t}\,dt$ satisfies $f'(x) = 2^x/x$ for all $x > 0$ and $f(1) = 0$.

5. The region R is shown to the right.



Using the shell method and integration by parts, we find that

$$V = \int_0^{\pi/2} 2\pi x \cos x \, dx \qquad \begin{array}{ll} u = x & dv = \cos x \, dx \\ du = dx & v = \sin x \end{array}$$

$$= 2\pi \left(x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx \right)$$

$$= 2\pi \left(\frac{\pi}{2} + \cos x \Big|_0^{\pi/2} \right) = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi(\pi - 2)$$

The volume of the solid is $\pi(\pi-2)$ cubic units.

6. By splitting up the sum, we recognize two geometric series. Hence

$$\sum_{n=1}^{\infty} \frac{3^{n+1} - 2^n}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^{n-1}} - \sum_{n=1}^{\infty} \frac{2^n}{4^{n-1}}$$

$$= \sum_{n=1}^{\infty} 12\left(\frac{3}{4}\right)^n - \sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^n$$

$$= \frac{9}{1 - 3/4} - \frac{2}{1 - 1/2}$$

$$= 36 - 4$$

$$= 32$$

7. We begin with the Root Test:

$$l = \lim_{n\to\infty} \sqrt[n]{\frac{5}{n3^n}|x-2|^n} = \lim_{n\to\infty} \frac{\sqrt[n]{5}}{\sqrt[n]{n}} \cdot \frac{|x-2|}{3} = \frac{|x-2|}{3}$$

We need $l < 1$ for convergence so $|x-2| < 3$, that is, the radius of convergence is 3. For the interval of convergence, we must check the endpoints

$$|x-2| < 3 \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$$

$x = -1$ gives $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{5}{n}$ which converges by the Alternating Series Test

$x = 5$ gives $\sum_{n=1}^{\infty} \frac{5}{n}$, a divergent p-series.

The interval of convergence is thus $[-1, 5)$.

Let $f(x) = \sum_{n=1}^{\infty} \frac{5}{n3^n}(x-2)^n$ for $-1 \le x < 5$. The equation $f^{(n)}(2) = n!\, c_n$ relates the derivatives evaluated at the center with the coefficients. In this case

$$f^{(20)}(2) = 20!\, c_{20} = 20! \cdot \frac{5}{20 \cdot 3^{20}} = \frac{5 \cdot 19!}{3^{20}}.$$

8. We will evaluate (i) and (iii).

(i) $\int x \sin x^2\, dx = -\frac{1}{2}\cos x^2 + C$

$u = x \quad dv = \sin x\, dx$

$du = dx \quad v = -\cos x$

(iii) $\int x \sin x\, dx = -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + C$

9. We will use trig substitution.

$$x = \sec\theta \qquad\qquad x = 1 \Rightarrow \theta = 0$$
$$dx = \sec\theta\tan\theta\, d\theta \qquad\qquad x = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$$
$$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$$

$$\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x^4}\, dx = \int_0^{\pi/4} \frac{\tan\theta}{\sec^4\theta} \sec\theta\tan\theta\, d\theta$$
$$= \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^3\theta}\, d\theta = \int_0^{\pi/4} \sin^2\theta\cos\theta\, d\theta$$
$$= \frac{1}{3}\sin^3\theta \Big|_0^{\pi/4} = \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\sqrt{2}}{12}.$$

The value of the integral is $\frac{\sqrt{2}}{12}$.

10. The limit of the sequence $\{k(\sqrt[k]{10} - 1)\}$ is the same as the limit of $f(x) = x(10^{1/x} - 1)$ as $x \to \infty$.

$$\lim_{x\to\infty} x(10^{1/x} - 1) = \lim_{x\to\infty} \frac{10^{1/x} - 1}{1/x} \qquad \frac{0}{0} \text{ form}$$
$$= \lim_{x\to\infty} \frac{\ln 10 \cdot 10^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \qquad \text{L'Hopital's Rule}$$
$$= \lim_{x\to\infty} \ln 10 \cdot 10^{1/x} = \ln 10$$

The limit of the sequence is $\ln 10$.

$$u = e^{2x}$$

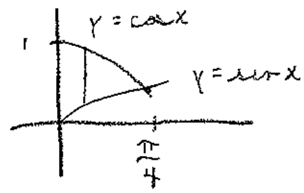
11. $$\int e^{2x}\tan^2 e^{2x}\, dx = \frac{1}{2}\int \tan^2 u\, du$$
$$= \frac{1}{2}\int (\sec^2 u - 1)\, du \qquad \text{trig identity}$$
$$= \frac{1}{2}(\tan u - u) + C$$
$$= \frac{1}{2}\left(\tan e^{2x} - e^{2x}\right) + C \qquad \text{return to } x$$

12. Since $0 \leq 1 - \sin^2 x = \cos^2 x < 1$ for $x \in (0, \pi)$, the series is a convergent geometric series

$$f(x) = \sum_{k=1}^{\infty} (1 - \sin^2 x)^k = \sum_{k=1}^{\infty} (\cos^2 x)^k = \frac{\cos^2 x}{1 - \cos^2 x} = \cot^2 x$$

It follows that $f\left(\frac{2\pi}{3}\right) = \cot^2\left(\frac{2\pi}{3}\right) = \frac{1}{3}$.

13. The base appears to the right



The length of a side of the square is $\cos x - \sin x$.

$$V = \int_0^{\pi/4} (\cos x - \sin x)^2 \, dx = \int_0^{\pi/4} (\cos^2 x - 2 \sin x \cos x + \sin^2 x) \, dx$$

$$= \int_0^{\pi/4} (1 - 2 \sin x \cos x) \, dx = (x - \sin^2 x) \Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$$

The volume of the solid is $\frac{\pi - 2}{4}$ cubic units.

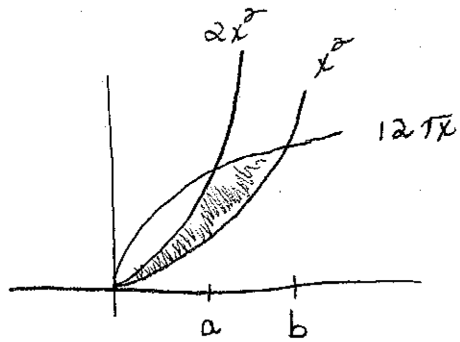
14. Given $\int_0^{2x} f(t)\,dt = 4 \sin x + x$, the FTC yields

$f(2x) \cdot 2 = 4 \cos x + 1$ when we differentiate both sides of the equation. Hence $f(2x) = 2 \cos x + \frac{1}{2}$.

When $x = \pi/6$ we obtain

$$f(\pi/3) = 2 \cos \frac{\pi}{6} + \frac{1}{2} = \sqrt{3} + \frac{1}{2}.$$

15. See the graph.



$2a^2 = 12\sqrt{a}$ $\qquad$ $b^2 = 12\sqrt{b}$

$a^{3/2} = 6$ $\qquad$ $b^{3/2} = 12$

$$A = \int_0^a (2x^2 - x^2)\,dx + \int_a^b (12\sqrt{x} - x^2)\,dx$$

$$= \tfrac{1}{3}x^3 \Big|_0^a + \left(8x^{3/2} - \tfrac{1}{3}x^3\right)\Big|_a^b$$

$$= \tfrac{1}{3}a^3 + \left(8b^{3/2} - \tfrac{1}{3}b^3\right) - \left(8a^{3/2} - \tfrac{1}{3}a^3\right)$$

$$= \tfrac{1}{3}\cdot 36 + \left(8\cdot 12 - \tfrac{1}{3}\cdot 144\right) - \left(8\cdot 6 - \tfrac{1}{3}\cdot 36\right)$$

$$= 12 + 48 - 36 = 24$$

The area of the shaded region is 24 square units.

16. $\displaystyle\int_0^{\pi} \frac{\sin t}{(3+2\cos t)^3}\,dt = \int_5^1 \frac{1}{u^3}\left(-\tfrac{1}{2}\,du\right)$ $\qquad$ $u = 3 + 2\cos t$, $du = -2\sin t\,dt$

$$= \tfrac{1}{2}\int_1^5 u^{-3}\,du$$

$$= -\tfrac{1}{4}u^{-2}\Big|_1^5$$

$$= -\tfrac{1}{4}\left(\tfrac{1}{25} - 1\right)$$

$$= \frac{6}{25}.$$

17. We use partial fractions

$$\frac{2x+3}{x^2-4x-5} = \frac{2x+3}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

$$2x+3 = A(x+1) + B(x-5)$$

$$x = 5 \Rightarrow 13 = 6A \text{ so } A = \frac{13}{6}$$

$$x = -1 \Rightarrow 1 = -6B \text{ so } B = -\frac{1}{6}$$

$$\int \frac{2x+3}{x^2-4x-5}\,dx = \int \left(\frac{13/6}{x-5} - \frac{1/6}{x+1} \right) dx$$

$$= \frac{13}{6} \ln|x-5| - \frac{1}{6} \ln|x+1| + C$$

18. This is an improper integral so we use limits

$$\int_1^\infty \frac{4x-3}{x^4}\,dx = \lim_{b\to\infty} \int_1^b \left(\frac{4}{x^3} - \frac{3}{x^4} \right) dx$$

$$= \lim_{b\to\infty} \left(-\frac{2}{x^2} + \frac{1}{x^3} \right) \Big|_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{2}{b^2} + \frac{1}{b^3} - (-1) \right)$$

$$= 1$$

19. Note that $\left(\frac{n}{n+2}\right)^n = \left(\frac{n+2}{n}\right)^{-n} = \left[\left(1+\frac{2}{n}\right)^n\right]^{-1}$

Since $\left\{\left(1+\frac{a}{n}\right)^n\right\}$ converges to e^a, the sequence

$\left\{\left(\frac{n}{n+2}\right)^n\right\}$ converges to $(e^2)^{-1} = e^{-2}$.

20. We must find the first three derivatives:

$$\begin{array}{lll} f(x) = \sqrt{1+x} & f(3) = 2 & 0! \\ f'(x) = \frac{1}{2}(1+x)^{-1/2} & f'(3) = \frac{1}{4} & 1! \\ f''(x) = -\frac{1}{4}(1+x)^{-3/2} & f''(3) = -\frac{1}{32} & 2! \\ f'''(x) = \frac{3}{8}(1+x)^{-5/2} & f'''(3) = \frac{3}{256} & 3! \end{array}$$

The Taylor series for $\sqrt{1+x}$ at $x = 3$ is

$$2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 + \cdots$$

21. The total distance traveled is $\int_0^4 |v(t)|\,dt$.

$$v(t) = 0 \Rightarrow t(5-t^2) = 0 \Rightarrow t = 0, \pm\sqrt{5}$$

$$\begin{aligned} \int_0^4 |v(t)|\,dt &= \int_0^{\sqrt{5}} (5t - t^3)\,dt + \int_{\sqrt{5}}^4 (t^3 - 5t)\,dt \\ &= \left(\tfrac{5}{2}t^2 - \tfrac{1}{4}t^4\right)\Big|_0^{\sqrt{5}} + \left(\tfrac{1}{4}t^4 - \tfrac{5}{2}t^2\right)\Big|_{\sqrt{5}}^4 \\ &= \left(\tfrac{25}{2} - \tfrac{25}{4}\right) + (64 - 40) - \left(\tfrac{25}{4} - \tfrac{25}{2}\right) \\ &= \tfrac{25}{4} + 24 + \tfrac{25}{4} \\ &= \tfrac{25}{2} + 24 \\ &= 36\tfrac{1}{2} \end{aligned}$$

The particle travels $36\frac{1}{2}$ meters during this time period.

22. We first find an equation for the parabola. With the origin at the midpoint of the base and feet as the units, the equation has the form $y = 1 - kx^2$, where k is a constant.

Since $y = \frac{2}{3}$ when $x = 10$, we find that

$\frac{2}{3} = 1 - 100k \Rightarrow k = \frac{1}{300}$. The cross-sectional area of the roadbed is

$$2\int_0^{10}\left(1 - \frac{1}{300}x^2\right)dx = 2\left(x - \frac{1}{900}x^3\right)\Big|_0^{10}$$

$$= 2\left(10 - \frac{10}{9}\right)$$

$$= \frac{160}{9}$$

The volume of crushed rock is $\frac{160}{9} \cdot 5400$ cubic feet. Since there are 27 cubic feet in a cubic yard, we need a total of

$$\frac{160}{9} \cdot 5400 \cdot \frac{1}{27} = \frac{160 \cdot 200}{9} = \frac{32000}{9} = 3555\tfrac{5}{9}$$

cubic yards of crushed rock.

23. We use the Ratio Test for the first series. Since

$$l = \lim_{k\to\infty}\left(\frac{5\cdot 8\cdot 11\cdots(3k+5)}{2^{k+1}(k+1)!}\cdot\frac{2^k k!}{5\cdot 8\cdot 11\cdots(3k+2)}\right)$$

$$= \lim_{k\to\infty}\frac{3k+5}{2(k+1)} = \frac{3}{2} > 1,$$

the series diverges.

For the second series, note that $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}(3k^2+1)}{k^4+8k-3}\right|$ is similar to $\sum_{k=1}^{\infty}\frac{1}{k^2}$, a convergent p-series. Since

$$\lim_{k\to\infty}\frac{\left|\frac{(-1)^{k+1}(3k^2+1)}{k^4+8k-3}\right|}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{3k^4+k^2}{k^4+8k-3} = 3,$$

the original series converges absolutely by the Limit Comparison Test.

Since the sequence $\left\{\frac{1}{4k-1}\right\}$ decreases to 0, the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k-1}$ converges by the Alternating Series Test.

Note that $\lim_{k\to\infty}\frac{1}{\sqrt[k]{4k}} = \lim_{k\to\infty}\left(\frac{1}{\sqrt[k]{4}}\cdot\frac{1}{\sqrt[k]{k}}\right) = 1 \neq 0.$

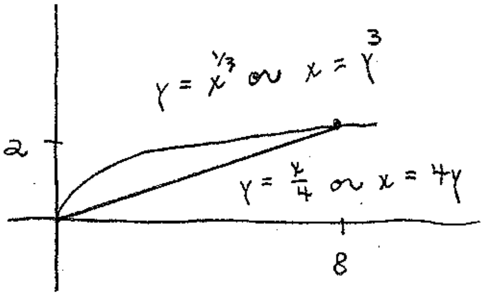
Hence, the series diverges by the Divergence Test.

For the last series, we use the Root Test. Since

$$l = \lim_{k\to\infty}\sqrt[k]{\left(\frac{k}{3k+1}\right)^k} = \lim_{k\to\infty}\frac{k}{3k+1} = \frac{1}{3} < 1,$$

the series converges.

24. We first sketch the region R.



a) $V = \int_0^8 \left(\pi x^{2/3} - \pi \cdot \frac{x^2}{16}\right) dx$ using washers

$V = \int_0^2 2\pi y \left(4y - y^3\right) dy$ using shells

b) $V = \int_0^8 2\pi x \left(x^{1/3} - \frac{x}{4}\right) dx$ using shells

$V = \int_0^2 \left(\pi \cdot 16y^2 - \pi y^6\right) dy$ using washers

c) $V = \int_0^8 2\pi(8-x)\left(x^{1/3} - \frac{x}{4}\right) dx$ using shells

d) $V = \int_0^2 \left(4y - y^3\right)^2 dy$