

A Brief Summary of Math 225

You should know about curves in **parametric form** and how to parameterize simple curves such as circles. Given a curve in this form, you should be able to find its tangent lines, the area beneath it, and its arc length. You should be familiar with **polar coordinates** and the relationship between (x, y) coordinates and (r, θ) coordinates. Know how to graph simple curves in polar coordinates and how to find the area of regions bounded by polar curves. Other than knowing what the basic **conic sections** are (**parabolas**, **circles**, **ellipses**, and **hyperbolas**) we do not focus on any of their special forms or properties.

You should be able to work with **vectors** in \mathbb{R}^2 and \mathbb{R}^3 , including how to represent them, perform the operations of addition and scalar multiplication, and find their magnitude. The **dot product** of two vectors is useful for finding the angle between two vectors (and thus determining when two vectors are **orthogonal**), while the **cross product** of two vectors in \mathbb{R}^3 generates a vector that is orthogonal to each of the original vectors. Equations of **lines** and **planes** have various representations in \mathbb{R}^3 and you should be able to work with these in various settings. Finally, you need to be familiar with the basic ideas behind **cylindrical coordinates** and **spherical coordinates**.

You should know a few general things about functions of several variables such as the nature of their graphs, difficulties that can arise when computing limits, and the concepts of **level curves** and **level surfaces**. You can then move on to **partial derivatives** and their significance. **Clairaut's Theorem** states that mixed partial derivatives are equal under certain conditions of continuity. The **tangent plane** (which gives a linear approximation for the function) to the level surface $f(x, y, z) = k$ at a point (x_0, y_0, z_0) is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The **Chain Rule** for functions of several variables is messy but you should have some idea of how it works. You should know what the **gradient vector**, denoted ∇f , represents and how it is useful for computing **directional derivatives**. (Remember that you need a unit vector when computing directional derivatives.) Finding **max/min** values for functions of two or more variables requires some care and the **Second Derivative Test** is more involved (recall that saddle points may occur). If the problem involves a closed and bounded region, the max/min values may occur on the boundary but, since the boundary consists of a curve or a surface rather than two endpoints, more effort is required. You should also be familiar with the basic idea behind **Lagrange multipliers** for finding the extreme values of a function subject to a constraint.

You should be familiar with the basic ideas behind the concepts of **double integrals** and **triple integrals**, including **iterated integrals** and **Fubini's Theorem**. You should be able to set up these integrals and change the order of integration when necessary. Double integrals in polar coordinates involve the $r \, dr \, d\theta$ term and sometimes make an integral much easier to evaluate. Triple integrals in cylindrical coordinates involve the term $r \, dz \, dr \, d\theta$, while those in spherical coordinates involve the term $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$. Applications of these integrals include area, volume, mass, and center of mass. The area of the surface defined by $z = f(x, y)$ for (x, y) in the region D is given by

$$\iint_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2} \, dA.$$

We do not expect you to know the change of variables formula (involving **Jacobians**) for multiple integrals.

You should know what **vector fields** are and how to visualize them in \mathbb{R}^2 and \mathbb{R}^3 . A vector field \mathbf{F} is **conservative** if there exists a function f such that $\mathbf{F} = \nabla f$.

The **line integral** (with respect to arc length) of a function f (we give the \mathbb{R}^3 version; \mathbb{R}^2 is similar) along a curve C represented parametrically by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is defined by

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

The line integral of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ along C is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} \equiv \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C (P dx + Q dy + R dz).$$

One interpretation of this integral is the work done by a force moving an object along the curve C .

A line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if its value depends only on the endpoints of C , not on the path. It then follows that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path. A vector field with this property must be conservative (as defined above). We thus obtain the **Fundamental**

Theorem for Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. You should know how to check that a vector field \mathbf{F} is conservative and, if it is, how to find a function f so that $\mathbf{F} = \nabla f$. Recognizing this can make the evaluation of some line integrals much easier.

Green's Theorem states that $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} (P dx + Q dy)$, where D is a region in the plane and ∂D is its positively oriented closed boundary curve. For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, this theorem becomes

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T} ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad \text{and} \quad \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA.$$

One simple consequence of this theorem is a method to find the area A of the region D :

$$A = \oint_{\partial D} x dy = - \oint_{\partial D} y dx = \frac{1}{2} \oint_{\partial D} (x dy - y dx).$$

The **curl** of a vector field \mathbf{F} is the vector field defined by $\nabla \times \mathbf{F}$ and the **divergence** of a vector field \mathbf{F} is the scalar $\nabla \cdot \mathbf{F}$, where ∇ is a symbolic notation for the operator $\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$. Under appropriate hypotheses, a vector field is conservative if and only if $\text{curl } \mathbf{F} = \mathbf{0}$.

(The following are not part of the written exam but are included for completeness.)

If a surface M is given by $z = g(x, y)$ for (x, y) in a domain D , then the **surface integral** of f on M is

$$\iint_M f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + (g_x(x, y))^2 + (g_y(x, y))^2} dA.$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, then the **flux** of \mathbf{F} across an oriented surface M with normal \mathbf{n} is

$$\iint_M \mathbf{F} \cdot \mathbf{n} dS = \iint_D (-Pg_x - Qg_y + R) dA.$$

Stokes' Theorem states that $\iint_M \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \int_{\partial M} \mathbf{F} \cdot \mathbf{T} ds$, where M is a surface and ∂M is its closed boundary curve. This theorem shows that the flux of the curl over a surface equals the circulation of the function around the boundary. Sometimes one of these integrals is much easier to evaluate than the other.

The **Divergence Theorem** states that $\iiint_E \text{div } \mathbf{F} dV = \iint_{\partial E} \mathbf{F} \cdot \mathbf{n} dS$, where E is a solid and ∂E is its boundary surface. This theorem shows that the sum of all the divergences over the solid equals the flux through the boundary surface. Sometimes one of these integrals is much easier to evaluate than the other.