1. We first determine the plane containing the points A, B, and C. To find the normal vector, we compute

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, 1, 2\rangle \times \langle -1, 4, -2\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 4 & -2 \end{vmatrix} = \langle -10, 0, 5\rangle$$

A normal vector to the plane is thus $\langle 2, 0, -1\rangle$ and an equation for the plane is $2(x-1) - (z-1) = 0$ or $2x - z = 1$. The line through D perpendicular to the plane has parametric equations

$$\begin{aligned} x &= 3 + 2t \\ y &= 4 \\ z &= 10 - t \end{aligned}$$

and meets the plane when

$$\begin{aligned} 2(3+2t) - (10-t) &= 1 \\ 5t - 4 &= 1 \\ t &= 1 \end{aligned}$$

The point $(5, 4, 9)$ is the point on the plane that is closest to D.

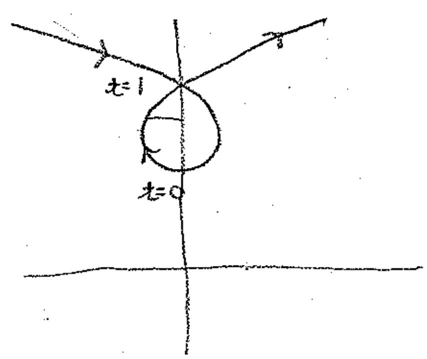
2. For part (a), the point $(6, 12)$ corresponds to $t = 2$ and

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{dy/dt}{dx/dt}\right|_{t=2} = \left.\frac{2t}{3t^2-1}\right|_{t=2} = \frac{4}{11}$$

An equation for the tangent line is $y - 12 = \frac{4}{11}(x-6)$.

Using a table of values, we can sketch the curve.

t	(x,y)
-3	$(-24,17)$
-2	$(-6,12)$
-1	$(0,9)$
0	$(0,8)$
1	$(0,9)$
2	$(6,12)$
3	$(24,1$

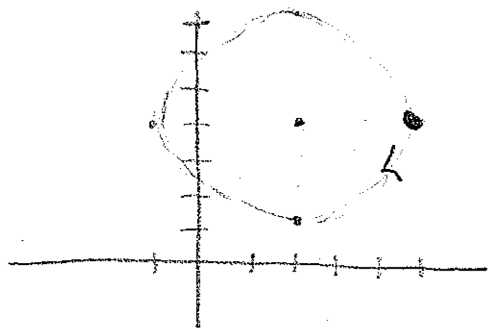


$$A = 2\int_{t=0}^{t=1} -x\,dy = 2\int_0^1 (t-t^3)\,2t\,dt = 4\int_0^1 (t^2 - t^4)\,dt$$

$$= 4\left(\frac{1}{3}t^3 - \frac{1}{5}t^5\right)\Big|_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}.$$

The area of the loop is $4/15$ square units.

3. The center of the circle is $(2,4)$ and the radius is 3.



The parametric equations are

$$x = 2 + 3\cos(2\pi t)$$
$$y = 4 - 3\sin(2\pi t)$$

4. Since $z = x^2 + 2y^3 - 4xy$, we have

$$\frac{\partial z}{\partial x} = 2x - 4y \qquad \text{and} \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,-1)} = 6$$

$$\frac{\partial z}{\partial y} = 6y^2 - 4x \qquad\qquad \left.\frac{\partial z}{\partial y}\right|_{(1,-1)} = 2$$

A normal for the tangent plane is $\langle 6, 2, -1\rangle$ so an equation for the plane is $6(x-1) + 2(y+1) - (z-3) = 0$ or

$$6x + 2y - z = 1.$$

5. We first find the gradient of f at $(2,4,1)$

$$f_x(x,y,z) = \frac{1}{3z-y} \qquad f_x(2,4,1) = -1$$

$$f_y(x,y,z) = \frac{(3z-y)2 - (x+2y)(-1)}{(3z-y)^2} = \frac{6z+x}{(3z-y)^2} \qquad f_y(2,4,1) = 8$$

$$f_z(x,y,z) = \frac{-3(x+2y)}{(3z-y)^2} \qquad f_z(2,4,1) = -30$$

Thus $\nabla f(2,4,1) = \langle -1, 8, -30\rangle$. A unit vector in the given direction is $\bar{u} = \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle$. It follows that

$$D_{\bar{u}} f(2,4,1) = \nabla f(2,4,1) \cdot \bar{u} = \frac{37}{\sqrt{3}}.$$

6. The larger graph corresponds to $r = 4\cos\theta$. The curves meet when $16\cos^2\theta = 8\cos\theta$ or $8\cos\theta(2\cos\theta - 1) = 0$ so θ is $\frac{\pi}{2}$ or $\frac{\pi}{3}$. Taking advantage of symmetry,

$$A = 2\int_0^{\pi/3}\left(\tfrac{1}{2}(4\cos\theta)^2 - \tfrac{1}{2}\left(\sqrt{8\cos\theta}\right)^2\right)d\theta$$

$$= \int_0^{\pi/3}(16\cos^2\theta - 8\cos\theta)\,d\theta$$

$$= \int_0^{\pi/3}(8 + 8\cos 2\theta - 8\cos\theta)\,d\theta$$

$$= 8\left(\theta + \tfrac{1}{2}\sin 2\theta - \sin\theta\right)\Big|_0^{\pi/3}$$

$$= 8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right) = \frac{8\pi}{3} - 2\sqrt{3}$$

The area of the region is $\frac{8\pi}{3} - 2\sqrt{3}$ square units.

7. The line segment can be represented parametrically by

$$x = 4t \qquad \qquad \frac{dx}{dt} = 4$$
$$y = 1 + t \quad \text{for } 0 \le t \le 1. \text{ Note that } \quad \frac{dy}{dt} = 1$$
$$z = 2 + t \qquad \qquad \frac{dz}{dt} = 1$$

It follows that

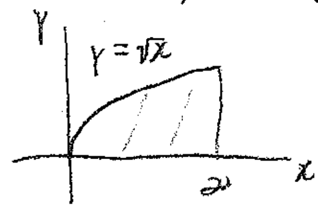
$$\int_C xy\,ds = \int_0^1 4t(1+t)\sqrt{18}\,dt$$

$$= 12\sqrt{2}\int_0^1 (t + t^2)\,dt$$

$$= 12\sqrt{2}\left(\tfrac{1}{2} + \tfrac{1}{3}\right)$$

$$= 10\sqrt{2}.$$

8. Since $z = 4 - x^2$ meets $z = 0$ when $x = 2$, the part of the region in the xy plane is



It follows that

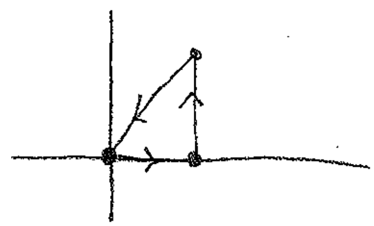
$$\iiint_S 3xy\, dV = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} 3xy\, dz\, dy\, dx$$

$$= \int_0^2 3x \cdot \tfrac{1}{2}x \cdot (4 - x^2)\, dx$$

$$= \frac{3}{2}\int_0^2 (4x^2 - x^4)\, dx = \frac{3}{2}\left(\frac{4}{3}\cdot 8 - \frac{1}{5}\cdot 32\right)$$

$$= 16\left(1 - \frac{3}{5}\right) = \frac{32}{5}.$$

9. Since C is closed (see the graph), we can use Green's Theorem.



$$\int_C \left(\sqrt{1+x^3}\, dx + 2xy\, dy\right) = \int_0^1 \int_0^{3x} 2y\, dy\, dx$$

$$= \int_0^1 9x^2\, dx$$

$$= 3$$

10. By inspection, we see that $\bar{F} = \nabla f$ for the function $f(x,y,z) = xyz + \frac{1}{2}z^2$. Note that the curve begins at $(1,0,0)$ and ends at $(-1,0,7)$. By the fundamental theorem for line integrals

$$\int_C \bar{F} \cdot d\bar{r} = f(-1,0,7) - f(1,0,0) = \frac{49}{2}.$$

The work done by the force is $\frac{49}{2}$ units.

11. The paraboloid meets the xy-plane in the circle $x^2+y^2=4$ so cylindrical coordinates are a good choice.

$$\text{mass} = \iiint_S \rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_0^{8-2r^2} 3r^2\cos^2\theta \cdot r\,dz\,dr\,d\theta$$

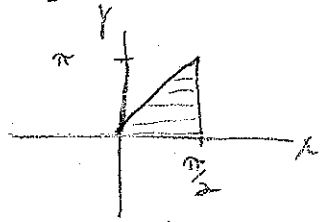
$$= \int_0^2 3r^3(8-2r^2)\,dr \cdot \int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 6\pi\int_0^2 (4r^3 - r^5)\,dr$$

$$= 6\pi\left(16 - \frac{64}{6}\right) = 32\pi$$

The mass of the solid is 32π units.

12. Given the integrand, we need to change the order of integration



$$\int_0^{\pi}\int_{y/2}^{\pi/2} \frac{\sin x}{x}\,dx\,dy = \int_0^{\pi/2}\int_0^{2x} \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^{\pi/2} 2\sin x\,dx$$

$$= -2\cos x\Big|_0^{\pi/2}$$

$$= 2$$

13. Both parts require the gradient of T.

$$T_x(x,y) = \frac{-x}{\sqrt{20-x^2-7y^2}} \qquad T_x(2,1) = -\frac{2}{3}$$

$$T_y(x,y) = \frac{-7y}{\sqrt{20-x^2-7y^2}} \qquad T_y(2,1) = -\frac{7}{3}$$

It follows that $\nabla T(2,1) = -\frac{1}{3}\langle 2,7\rangle$.

A unit vector from $(2,1)$ to $(0,0)$ is $\bar{u} = -\frac{1}{\sqrt{5}}\langle 2,1\rangle$

$$D_{\bar{u}}T(2,1) = \nabla T(2,1)\cdot\bar{u} = \frac{11}{3\sqrt{5}}$$

The rate of change toward the origin is $\frac{11}{3\sqrt{5}}$ °C per length.

The temperature decreases most rapidly in the direction $-\nabla T$.

A unit vector in this direction is $\frac{1}{\sqrt{53}}\langle 2,7\rangle$.

14. The point $(1,2,0)$ is on the line and the direction of the line is $\langle -1,1,1\rangle$. The vector from $(1,2,0)$ to $(1,0,1)$ is $\langle 0,-2,1\rangle$. A normal vector for the plane is

$$\langle -1,1,1\rangle \times \langle 0,-2,1\rangle = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \langle 3,1,2\rangle$$

An equation for the plane is $3x + y + 2z = 5$.

15. We need to evaluate $\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$,

where C_1 is $\begin{aligned} x &= t \\ y &= t \\ z &= 0 \\ 0 \le\ &t \le 1 \end{aligned}$ and C_2 is $\begin{aligned} x &= 1 \\ y &= 1 \\ z &= t \\ 0 \le\ &t \le 1 \end{aligned}$

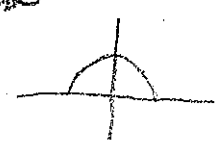
$$\int_{C_i} \bar{F}\cdot d\bar{r} = \int_{C_i} (y\,dx + x\,dy + xyz\,dz)$$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^1 (t + t + 0)\,dt = t^2\Big|_0^1 = 1$$

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_0^1 t\,dt = \tfrac{1}{2}$$

The work done by the force is $3/2$ units.

16. Given the form of the integral, we should convert to polar coordinates



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{\sqrt{2}} r\cdot r\,dr\,d\theta$$

$$= \pi\cdot \frac{1}{3} r^3 \Big|_0^{\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3}\pi$$

17. We will use Lagrange multipliers

maximize $f(x,y,z) = 2x - 2y + z$

subject to $g(x,y,z) = 0$ where $g(x,y,z) = x^2 + 2y^2 + 3z^2 - 114$

Set $\nabla f = \lambda \nabla g$ to obtain

$$\begin{aligned} 2 &= \lambda \cdot 2x \\ -2 &= \lambda \cdot 4y \\ 1 &= \lambda \cdot 6z \end{aligned} \quad \Rightarrow \quad \begin{aligned} \tfrac{1}{\lambda} &= x \\ \tfrac{1}{\lambda} &= -2y \\ \tfrac{1}{\lambda} &= 6z \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 6z \\ y &= -3z \end{aligned}$$

Using $g(x,y,z) = 0$ yields $36z^2 + 18z^2 + 3z^2 = 114$

so $z = \pm\sqrt{2}$. We clearly need $z = \sqrt{2}$. The maximum value of f is $12\sqrt{2} + 6\sqrt{2} + \sqrt{2} = 19\sqrt{2}$.

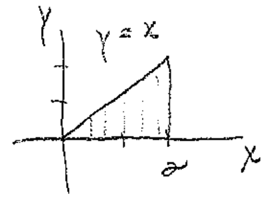
18. Given how messy this looks, we better hope that $\bar{F}$ is conservative. Indeed $\bar{F} = \nabla f$ for $f(x,y,z) = 3xy^3 - 5x^2z^2$. The initial point on the curve is $(1,3,3)$ and the terminal point is $(1,1,-1)$. Hence

reverse

$$\begin{aligned} \int_C \bar{F} \cdot d\bar{r} &= f(1,3,3) - f(1,1,-1) \\ &= (81 - 45) - (3 - 5) \\ &= -38 \end{aligned}$$

The work done by the force is -38 units.

19. Referring to the figure,



$$V = \int_0^2 \int_0^x (2x + 2y + 5)\,dy\,dx$$

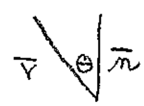
$$= \int_0^2 (2x^2 + x^2 + 5x)\,dx$$

$$= \left(x^3 + \tfrac{5}{2}x^2\right)\Big|_0^2$$

$$= 18$$

The volume of the solid is 18 cubic units.

20. The common normal for the planes is $\langle 2, -1, 2\rangle$; call this $\bar{n}$. Points $(0, -4, 0)$ and $(0, -13, 0)$ are on each plane and $\bar{v} = \langle 0, -9, 0\rangle$ is the vector joining them.



The distance between the planes is

$$\|\text{proj}_{\bar{v}}\, \bar{n}\| = \big|\|\bar{v}\| \cos\theta\big| = \left|\|\bar{v}\| \cdot \frac{\bar{v}\cdot\bar{n}}{\|\bar{v}\|\,\|\bar{n}\|}\right|$$

$$= \frac{|\bar{v}\cdot\bar{n}|}{\|\bar{n}\|} = \frac{9}{3} = 3.$$

The distance between the planes is 3 units.

21. The direction of the line is $\langle 6, -10, 2\rangle = -2\langle -3, 5, -1\rangle$ and the normal to the tangent plane is $\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\rangle$ or $\langle 2xy - 4y + 5, x^2 - 4x, -1\rangle$. We thus need $x^2 - 4x = 5$ so $x = -1$ is one option. Then $-2y - 4y + 5$ must be -3, which gives $y = 4/3$, and thus

$$z = \frac{4}{3} + \frac{16}{3} - 5 - 8 = -\frac{19}{3}.$$

The tangent plane at the point $(-1, 4/3, -19/3)$ is perpendicular to the given line.

22. The max/min values occur on the boundary or at an interior critical point. Since

$T_x(x,y) = 2x-1$ and $T_y(x,y) = 4y$,

the only critical point is $(\frac{1}{2},0)$. At this point, the temperature is $T(\frac{1}{2},0) = 99.75$. The boundary of the plate is $\{(x,y): x^2+y^2=1\}$ and here we have

$T(x,y) = x^2+2(1-x^2)-x+100 = 102-x-x^2$

for $-1 \le x \le 1$. Completing the square,

$$102-x-x^2 = 102-(x^2+x+\tfrac{1}{4})+\tfrac{1}{4}$$
$$= 102.25-(x+\tfrac{1}{2})^2$$

The max occurs when $x=-\frac{1}{2}$ and the min occurs when $x=1$; these values are 102.25 and 100, respectively. Therefore, the minimum temperature is 99.75° C at the point $(\frac{1}{2},0)$ and the maximum temperature is 102.25° C at the points $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2},\frac{\sqrt{3}}{2})$.

23. The upper half of the circle can be expressed as

$x = 2\cos t$ for $0 \le t \le \pi$. Calling this curve C,
$y = 2\sin t$

we find that

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (-y^2dx+2x\,dy)$$
$$= \int_0^\pi (8\sin^3 t + 8\cos^2 t)\,dt$$
$$= 8\int_0^\pi \left((1-\cos^2 t)\sin t + \tfrac{1}{2}+\tfrac{1}{2}\cos 2t\right)dt$$
$$= 8\left(-\cos t+\tfrac{1}{3}\cos^3 t+\tfrac{1}{2}t+\tfrac{1}{4}\sin 2t\right)\Big|_0^\pi$$
$$= 8\left(\left(1-\tfrac{1}{3}+\tfrac{\pi}{2}\right)-\left(-1+\tfrac{1}{3}\right)\right) = 8\left(\tfrac{4}{3}+\tfrac{\pi}{2}\right)$$

The work done by the force is $\frac{32}{3}+4\pi$ units.

24. Since S is a hemisphere, it is a good idea to evaluate this integral using spherical coordinates

$$\iiint_S z\,dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2} \rho\cos\phi \cdot \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

$$= 2\pi \int_0^{\pi/2} \int_0^{2} \rho^3 \sin\phi \cos\phi \; d\rho\, d\phi$$

$$= 2\pi \int_0^{\pi/2} 4 \sin\phi \cos\phi \, d\phi$$

$$= 2\pi \left(2\sin^2\phi\right)\Big|_0^{\pi/2}$$

$$= 4\pi.$$

25. (As mentioned in the summary, this sort of problem does not appear on our written exams, but it is a good review.)

The Divergence Theorem states that

$$\iiint_E \operatorname{div} \bar{F}\, dV = \iint_{\partial E} \bar{F} \cdot \bar{n}\, dS$$

For our function and solid, we have

$$\iiint_E \operatorname{div} \bar{F} \cdot dV = \iiint_E 2\,dV$$

$$= 2 \cdot \text{volume of } E$$

$$= 2 \cdot \tfrac{1}{2} \cdot \tfrac{4}{3}\pi$$

$$= \tfrac{4}{3}\pi,$$

which is a very simple computation.

The boundary of E consists of two surfaces

M_1 is the top half of the sphere

M_2 is the unit disk in the xy-plane, call this D

$$\iint_{\partial E} \bar{F}\cdot\bar{n}\,dS = \iint_{M_1} \bar{F}\cdot\bar{n}\,dS + \iint_{M_2} \bar{F}\cdot\bar{n}\,dS$$

$$= \iint_D \langle x,y,y^2\rangle\cdot\langle x,y,z\rangle\,\frac{dA}{z} +$$

$$\iint_D \langle x,y,y^2\rangle\cdot\langle 0,0,-1\rangle\,dA$$

$$= \iint_D \left(\frac{x^2+y^2+y^2z}{z} - y^2\right) dA$$

$$= \iint_D \frac{x^2+y^2}{\sqrt{1-x^2-y^2}}\,dA$$

$$= \int_0^{2\pi}\int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr\,d\theta$$

$r = \sin t$

$$= 2\pi\int_0^{\pi/2} \sin^3 t\,dt$$

$$= 2\pi\int_0^{\pi/2} (1-\cos^2 t)\sin t\,dt$$

$$= 2\pi\left(-\cos t + \tfrac{1}{3}\cos^3 t\right)\Big|_0^{\pi/2}$$

$$= 2\pi\cdot\frac{2}{3}$$

$$= \frac{4}{3}\pi$$

This computation is much more involved, but the answers are the same as expected by the Divergence Theorem.