A Brief Summary of Math 240

Linear algebra begins with the simple idea of solving two equations in two unknowns and then extending this to more equations and unknowns. However, there are a number of deep ideas behind this process and it is essential that you be familiar with these concepts. For instance, solving a system of linear equations can be interpreted as working with vectors in $\mathbb{R}^n$. It can also be connected to matrix multiplication and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. These ideas can then be extended to arbitrary vector spaces, which hint at the level of abstraction that is expected for mathematics majors.

You should be able to solve a **system of linear equations**, including both the **homogeneous** and **nonhomogeneous** cases, representing the solutions with **parameters** or **free variables** as needed. You should realize that such systems have no solutions, unique solutions, or an infinite number of solutions and be able to decide when each of these situations occurs. You should be able to make conversions between systems of linear equations, vector equations, and matrix equations and how to interpret the solutions for each type. You should be able to reduce a matrix to **echelon form** and make conclusions based on its properties.

You should be able to perform operations on matrices, including **multiplication** (recall that multiplication is not commutative) and finding **transposes**. An $n \times n$ matrix A is **invertible** or **nonsingular** if there exists an $n \times n$ matrix B such that $AB = I = BA$; we usually denote this matrix as A^{-1}. You should be able to find the inverse of a matrix in simple cases. There are many ways to decide when a matrix is invertible and you should be familiar with these. One of these ways involves computing the **determinant** of a matrix.

You should know what a **vector space** is and be familiar with some common examples of vector spaces, including (but not limited to) $\mathbb{R}^n$, P_n (polynomials of degree $\leq n$), $M_{m\times n}$ ($m \times n$ matrices), and $\mathcal{C}([a,b])$ (the collection of continuous functions defined on $[a,b]$). A set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ of vectors in a vector space V is **linearly dependent** if and only if there exist scalars $c_1, c_2, \ldots, c_n$, not all of which are 0, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

In this case, one vector can be written as a **linear combination** of the others. The set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is linearly independent if it is not linearly dependent, that is, the set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is **linearly independent** if and only if the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. The **span** of $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is the set of all possible linear combinations of the vectors. A set that spans V and is linearly independent is known as a **basis** of V. The number of vectors in a basis is called the **dimension** of V.

A subset W of V that is itself a vector space is known as a **subspace** of V; you should know how to check that a set W is a subspace. An $m \times n$ matrix A determines several subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, namely the **column space** of A, the **row space** of A, and the **null space** of A. You should know how to find a basis for each of these subspaces. The dimension of the column space of A is known as the **rank** of A and the dimension of the null space of A is called the **nullity** of A. It is a theorem that the rank plus the nullity of an $m \times n$ matrix is n.

Let U and V be two arbitrary vector spaces. A function $T\colon U \to V$ is a **linear transformation** from U to V if $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all vectors $\mathbf{u}$ in U and $T(\mathbf{u_1} + \mathbf{u_2}) = T(\mathbf{u_1}) + T(\mathbf{u_2})$ for all vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ in U. The **range** and **kernel** of a linear transformation are defined by

$$\{T(\mathbf{u}) : \mathbf{u} \in U\} \quad \text{and} \quad \{\mathbf{u} \in U : T(\mathbf{u}) = \mathbf{0}\},$$

respectively. The range of T is a subspace of V and the kernel of T is a subspace of U; you should be able to prove these facts. An $m \times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ corresponds to multiplication by some matrix.

The number λ is an **eigenvalue** of a square matrix A if and only if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is then called an eigenvector for the matrix A. It may be helpful to think about the linear transformation defined by A as not changing the direction of the vector $\mathbf{x}$. The polynomial in λ defined by $\det(A - \lambda I)$ is known as the **characteristic polynomial** of A and (by finding the roots of the polynomial) can be used to find the eigenvalues of A. Once these values have been found, nontrivial solutions to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ give the corresponding eigenvectors.

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. The inner product $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$ and the **length** or **norm** of $\mathbf{u}$ is defined by $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$. (Each of these concepts has a simple formula in terms of the components of $\mathbf{u}$ and $\mathbf{v}$.) Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. More generally, the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ satisfies the equation $\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta = \mathbf{u} \cdot \mathbf{v}$; you should know that the law of cosines is behind this property of the inner product. You should be able to prove that a set of nonzero orthogonal vectors is linearly independent and be able to use the Gram-Schmidt process to convert a basis into an orthogonal basis.

More generally, in some cases, it is possible to define an inner product on other vectors spaces; you should know the properties that an inner product must satisfy. The standard notation becomes $\langle \mathbf{u}, \mathbf{v} \rangle$ in the general case. For instance, in the vector space $\mathcal{C}([a, b])$, an inner product is defined by $\langle f, g \rangle = \displaystyle\int_a^b f(t)g(t)\,dt$. This inner product plays a key role in the topic of Fourier series.

Two square matrices A and B are **similar** if there exists an invertible matrix P such that $A = PBP^{-1}$. Similar matrices have the same characteristic equation (you should be able to prove this) and thus the same eigenvalues. You should know what it means for a matrix to be similar to a diagonal matrix; the equation $AP = PD$ tells you a lot about the columns of A.

A square matrix A is **symmetric** if $A^T = A$. Symmetric matrices possess some special properties. For instance, all of the eigenvalues of a symmetric matrix are real and eigenvectors corresponding to distinct eigenvalues are orthogonal. It then follows (a proof is beyond the scope of the course) that A is orthogonally diagonalizable, that is, $AP = PD$, where the columns of P are orthonormal (so $P^{-1} = P^T$) and D is a diagonal matrix.