A Brief Summary of Math 260

You should know the five **logical connectives** and their corresponding **truth tables**. Given a specific conditional statement, you should be able to write its **converse** and **contrapositive**. The **existential quantifier** $\exists$ and the **universal quantifier** $\forall$ are very important concepts in higher level mathematics so you should be able to work with these, especially when they are mixed together in the same statement. Be familiar with **sets** and **set operations**. Know the definition of an **equivalence relation** and the properties of **equivalence classes**.

You should know the definition of **divisibility** for integers and be able to work with this concept, including verifying simple properties related to it. Know what it means for an integer to be **prime** or **composite**. Be able to compute the **greatest common divisor** of two integers and know what it means for two integers to be **relatively prime**. You should be able to prove simple results concerning prime numbers such as "if $p|(ab)$, then $p|a$ or $p|b$". You should be familiar with the spaces $\mathbb{Z}_n$ and $\mathbb{U}_n$ and their properties as well as be able to solve congruence equations.

You should understand **functions** at a deeper level, including terms such as **domain**, **codomain**, and **range**. You should be able to identify when a function is **injective**, **surjective**, or **bijective**. Be able to illustrate competence in working with **induced set functions**. Know what it means for a set to be **finite** or **infinite** as well as for a set to be **countably infinite** or **uncountable**.

Since one of the main goals of Math 260 is learning to write proofs, you should know various proof strategies such as **proof by induction** and **indirect proof** (recall that there are two types of indirect proofs) and be able to write proofs clearly, using complete sentences. You should be aware of various well-known results and, especially for those with simpler proofs, know how to prove them. For instance, you should be able to prove the following results.

- if $n|a$ and $n|b$, then $n|(ax + by)$ for all integers x and y
- the existence portion of the Fundamental Theorem of Arithmetic
- there are an infinite number of primes
- the number $\sqrt{n}$ is irrational for any positive integer n that is not a perfect square
- the composition of two injective (or surjective) functions is injective (or surjective)
- the set of rational numbers is countably infinite

You should be familiar with the following theorems or results. This means that you should know what they say, understand what they mean, and be able to use them to solve problems.

- the binomial theorem
- the division algorithm
- the Euclidean algorithm
- the Fundamental Theorem of Arithmetic
- Wilson's Theorem
- Euler's Theorem
- Fermat's Little Theorem

Intro to Higher Math questions

1. Find the multiplicative inverse of $[7]$ in the space $\mathbb{U}_{109}$. Carefully explain the ideas you use to obtain your answer.

2. Suppose that $a^2 + b^2 = c^2$, where a, b, and c are positive integers with no common factors. Prove that 7 does not divide c.

3. Use induction to prove that 9 divides $n^3 + (n+1)^3 + (n+2)^3$ for each positive integer n.

4. Let a, b, and c be integers. Prove that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.

5. Give specific examples to illustrate that the following statements are not true in the space $\mathbb{Z}_{221}$.

   i. If $[x] \cdot [y] = [0]$, then either $[x] = [0]$ or $[y] = [0]$.

   ii. If $[x] \cdot [y] = [1]$, then either $[x] = [-1]$ or $[x] = [1]$.

   iii. If $[x]^2 = [1]$, then either $[x] = [-1]$ or $[x] = [1]$.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = x^3$. Represent the set $f^{-1}\big([-1, 4]\big) \cap g\big((-2, -1]\big)$ using interval notation.

7. Define a relation on $\mathbb{R}$ by $x \sim y$ if and only if $x - y$ is a rational number. Prove that $\sim$ is an equivalence relation on $\mathbb{R}$ and list three elements of the equivalence class $\big[\sqrt{2}\,\big]$. You may use the elementary fact that the sum of two rational numbers is a rational number.

8. Determine the unique remainder when $89!$ is divided by 97. Show your work clearly (and as simply as possible) and explicitly mention results you are using.

9. Determine the unique remainder when 3^{100} is divided by 109. Show your work clearly (and as simply as possible) and explicitly mention results you are using.

10. Suppose that n divides both a and b, where n, a, and b are integers. Prove that n divides $ax + by$ for all integers x and y.

11. Prove that the number $\log_7 100$ is irrational.

12. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined recursively by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for all $n \geq 1$. Prove that 7 divides a_{4n} for all positive integers n.