Linear Algebra questions

1. Find the kernel of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x - 3y, x + z)$.

2. Find two linearly independent eigenvectors for the matrix $\begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

3. Let U and V be vector spaces and let $T\colon U \to V$ be a linear transformation. Prove that the kernel of T is a subspace of U.

4. Let A be an $n \times n$ matrix. Write down three qualitatively different statements that are equivalent to the statement "A is invertible".

5. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$.

6. Find all solutions to the system $\begin{aligned} x + 2y + 9z &= 3 \\ 2x - y + 4z &= 1 \\ -4x + 7y + 6z &= 3 \end{aligned}$.

7. Let $T\colon U \to V$ be a linear transformation of U onto V. Prove that the set $\{T(\mathbf{u_1}), \ldots, T(\mathbf{u_n})\}$ spans V if the set $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ spans U.

8. Recall that $\mathcal{C}^2([0,1])$ is the vector space of all real-valued functions whose second derivative is continuous on $[0, 1]$. Prove that $W = \{y \in \mathcal{C}^2([0,1]) : 2y'' + xy = 0\}$ is a subspace of $\mathcal{C}^2([0,1])$.

9. Suppose that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set of vectors in some vector space V. Determine whether or not the set $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{x} + \mathbf{z}\}$ is linearly independent.

10. Find a basis for the null space of A, a basis for the column space of A, and a basis for the row space of A if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

11. Let A be an $m \times n$ matrix. Prove that the columns of A are linearly independent if and only if the matrix $A^T A$ is nonsingular.

12. Let A be a nonsingular $n \times n$ matrix and suppose that λ is an eigenvalue for A. Explain why $\lambda \neq 0$, then prove that λ^{-1} is an eigenvalue for A^{-1}.

13. Let A be a symmetric matrix and suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A corresponding to distinct eigenvalues. Prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

14. An $n \times n$ matrix A is said to be skew–symmetric if $A^T = -A$. Show that the set of all $n \times n$ skew–symmetric matrices is a subspace of the set of all $n \times n$ matrices, then find a basis for the set of all 3×3 skew–symmetric matrices.

15. Let V be an inner product space and let $\{\mathbf{u}, \mathbf{v}\}$ be an orthogonal set of nonzero vectors in V. Prove that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set.

16. Each of the following statement is false. Find the simplest change that will make the statement true.

    (a) A set of m vectors in $\mathbb{R}^n$ with $m > n$ is linearly independent.

    (b) If A is an $m \times n$ matrix, then the matrix $A^T A$ is an $m \times m$ matrix.

    (c) Similar matrices have the same eigenvectors.

    (d) An orthogonal set of vectors in linearly independent.

17. Let A and B be $n \times n$ matrices and suppose that A is similar to B. Prove that A and B have the same eigenvalues.

18. Let A be a 24×60 matrix and suppose that the rows of A are linearly independent. What is the dimension of the null space of A? Explain.

19. Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix. Suppose that $AB = \Theta$ (the matrix of all 0's). Prove that the column space of B is a subset of the null space of A.

20. Define $T\colon M_{2\times 2} \to P_1$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)t + b$. Prove that T is a linear transformation, then find a basis for the kernel of T.

21. Determine whether or not the set $\{t^2 - t + 3, -2t^2 + 4t, t^2 + 5t - 6\}$ is a basis for P_2.

22. Let V and W be vector spaces and let $T\colon V \to W$ be a one-to-one linear transformation. Prove that T maps linearly independent subsets of V into linearly independent subsets of W.

23. Let A be an $n \times n$ invertible matrix and let B be an $n \times p$ matrix. Prove that the matrices B and AB have the same rank.

24. Find all values of a and b for which the following system is consistent.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= a \\ x_1 + x_2 + 2x_3 &= 1 \\ x_2 + x_3 &= b \end{aligned}$$