Some notes for Section 4.6

We begin by making a few brief comments about the following theorem. Recall that a function is said to be entire if it is analytic on $\mathbb{C}$, that is, it is differentiable at every complex number.

Liouville's Theorem: If $f(z)$ is a bounded entire function, then there exists a constant α such that $f(z) = \alpha$ for all $z \in \mathbb{C}$.

Consider the following list of real-valued functions:

$$\frac{x}{1+x^2}, \quad \frac{2x^2}{1+x^2}, \quad \frac{5x^4+x^3+1}{x^4-2x^2+3}, \quad \sin x, \quad \cos x \quad \frac{\sin x}{5+\cos x}, \quad \arctan x, \quad e^{-x^2}.$$

Each of these functions is infinitely differentiable on $\mathbb{R}$ and each of these functions is bounded on $\mathbb{R}$. Liouville's Theorem states that the only such complex valued functions are the constant functions. As noted in the textbook, this theorem provides a simple way to prove the Fundamental Theorem of Algebra. (By the way, can you show directly why each of the above functions is unbounded if x is replaced by z?)

Returning for a moment to basic differential calculus, suppose that f is differentiable on some interval $[a, b]$. In order to find the maximum value of f, we first solve the equation $f'(x) = 0$ to obtain the critical points that are in the interval (a, b). It then follows that the maximum value of f is the value of f at one of these critical points or at one of the endpoints. As a reminder, we present a very simple example.

Problem: Find the maximum and minimum outputs of $f(x) = 6x - x^3$ on the interval $[1, 2]$.

Solution: It is easy to verify that f' is zero when $x = \pm\sqrt{2}$. These are the critical inputs of f, but only $\sqrt{2}$ is in the given interval. Evaluating f at the endpoints and the appropriate critical input yields $f(1) = 5$, $f(\sqrt{2}) = 4\sqrt{2} \approx 5.657$, $f(2) = 4$. The maximum output of f on $[1, 2]$ is thus $4\sqrt{2}$, which occurs when $x = \sqrt{2}$, and the minimum output of f on $[1, 2]$ is 4, which occurs when $x = 2$.

Note that the maximum value of f (as well as the maximum value of $|f|$) on $[1, 2]$ occurs at an interior point. Once again, this cannot happen for complex valued functions.

Maximum Modulus Principle: A function that is analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

For a closed interval $[a, b]$ on the real line, the boundary consists of the two points a and b. The boundary of a domain in the complex plane is much more complicated. You should pause for a moment and draw a bounded domain (recall that a domain is an open connected set) with at least two quasi-circular "holes" in it and note that the boundary consists of at least three different contours.

On the next page, we present another option for solving problems such as the one given as Example 1 in the textbook. For the record, you should fill in the details in the book's approach to obtain the function $11 - 2\cos(2t)$.

Example 1: Find the maximum value of $|z^2+3z-1|$ in the disk $|z| \le 1$.

Solution: By the Maximum Modulus Principle, the maximum value of this modulus will occur on the boundary, that is, at points where $|z|=1$. Letting $z=x+iy$, we know that $|z|=1$ when $x^2+y^2=1$. Using this fact, we find that

$$z^2+3z-1=(x^2-y^2+3x-1)+i(2xy+3y)=(2x^2+3x-2)+iy(2x+3).$$

It then follows that (be certain that you follow every step)

$$\begin{aligned}
|z^2+3z-1|^2 &= (2x^2+3x-2)^2+y^2(2x+3)^2 \\
&= \big(x(2x+3)-2\big)^2+(1-x^2)(2x+3)^2 \\
&= -4x(2x+3)+4+(2x+3)^2 \\
&= 4+(3+2x)\big(-4x+2x+3\big) \\
&= 4+(3+2x)(3-2x) \\
&= 13-4x^2.
\end{aligned}$$

For x in the interval $[-1,1]$, the maximum value of this function is 13 and this occurs when $x=0$. When $x=0$, it then follows that $y=\pm 1$ and thus $z=\pm i$. Hence, the maximum value of $|z^2+3z-1|$ in the disk $|z|\le 1$ is $\sqrt{13}$ and this occurs when $z=\pm i$.

Exercise 4: If $p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ is a polynomial and $\max|p(z)|=M$ for $|z|=1$, then $|a_k|\le M$ for $k=0,1,\ldots,n$.

Solution: By basic properties of polynomials, we know that $p^{(k)}(0)=k!a_k$ for each $0\le k\le n$. In addition, by Theorem 20 with $z_0=0$ and $R=1$, we find that $|p^{(k)}(0)|\le k!M$. Putting these two facts together shows that $|a_k|\le M$ for $k=0,1,\ldots,n$. ■

Exercise 8: If f is analytic in the annulus $1\le|z|\le 2$ with $|f(z)|\le 3$ for $|z|=1$ and $|f(z)|\le 12$ for $|z|=2$, then $|f(z)|\le 3|z|^2$ for all z in the annulus.

Solution: Let g be the function defined by $g(z)=f(z)/(3z^2)$ for all z in the given annulus. Note that g is analytic on the annulus. In addition, we find that

$$|g(z)|=\left|\frac{f(z)}{3z^2}\right|\le\frac{|f(z)|}{3}\le 1 \quad \text{for } |z|=1 \qquad \text{and} \qquad |g(z)|=\left|\frac{f(z)}{3z^2}\right|\le\frac{|f(z)|}{12}\le 1 \quad \text{for } |z|=2.$$

Since $|g(z)|$ is bounded by 1 on the boundary of the annulus, we know (by Theorem 24) that $|g(z)|\le 1$ for all z inside the annulus. It follows that $|f(z)|\le 3|z|^2$ for all z in the annulus. ■