Chapter 4 Summary

Definition: A smooth arc in the complex plane is the range of a one-to-one function $z(t)$ that has a continuous nonzero derivative on the interval $[a, b]$.

Definition: A contour is a finite sequence of directed smooth arcs.

Theorem: If f is continuous on the directed smooth arc γ, then $\displaystyle\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$. For contours, we add the integrals over each of the directed smooth arcs. Note that the value of the integral (which can often be obtained by finding an antiderivative) is independent of the parametrizations of the arcs.

ML Theorem: If f is continuous on the contour Γ and M is a number that satisfies $|f(z)| \le M$ for all z on Γ, then $\displaystyle\left|\int_\gamma f(z)\,dz\right| \le ML$, where L is the length of Γ.

TFAE Theorem: Suppose that f is continuous in a domain D. Then the following statements are equivalent:

i) the function f has an antiderivative in D;
ii) the value of $\int_\Gamma f(z)\,dz = 0$ for every closed contour in D;
iii) the value of $\int_\Gamma f(z)\,dz$ is independent of path, depending only on the initial and terminal points.

Definition: A domain D is simply connected if the interior of each loop in D contains only points of D.

Cauchy's Integral Theorem: If f is analytic in a simply connected domain D, then the three statements in the TFAE Theorem are all true.

Deformation Theorem: Let f be analytic in a domain D containing the loops Γ_1 and Γ_2. If these loops can be continuously deformed into one another in D, then $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

Cauchy's Integral Formula: Let Γ be a simple closed positively oriented contour. If f is analytic in a simply connected domain D containing Γ and z is any point inside Γ, then $\displaystyle f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw$.

Theorem: If $f = u + iv$ is analytic in a domain D, then f has derivatives of all orders in D and $f^{(k)}$ is analytic for each k. Consequently, the functions $u(x, y)$ and $v(x, y)$ have partial derivatives of all orders in D.

Cauchy's Integral Formula for Derivatives: If f is analytic inside and on the simple closed positively oriented contour Γ, then $\displaystyle f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$ for each point z inside Γ.

Theorem: Let f be analytic inside and on the circle $C_r = \{z : |z - z_0| = r\}$. If $|f(z)| \le M$ for all z on C_r, then $\displaystyle |f^{(n)}(z_0)| \le \frac{n!M}{r^n}$ for all $n \ge 0$.

Liouville's Theorem: The only bounded entire functions are constant functions.

Maximum Modulus Principle: A function that is analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

Theorem: If $u(x, y)$ is a harmonic function on a simply connected domain D, then there is an analytic function $f(z)$ for which $\operatorname{Re} f(z) = u(x, y)$ on D. Consequently, a function that is harmonic on a bounded simply connected domain and continuous up to and including the boundary attains its maximum and minimum on the boundary.