Some notes for Chapter 6

Theorem 1: (Cauchy's Residue Theorem) If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at points $z_1, z_2, \ldots, z_n$ inside Γ, then

$$\oint_\Gamma f(z)\,dz = 2\pi\,i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

We have seen various ways throughout the textbook to find the residue of a function f at a singularity. See Example 2 in Section 6.1 for a method that can be useful in some cases.

Here are a few examples to illustrate Cauchy's Residue Theorem. For the record, simple closed curves are assumed to be traversed once in the positive direction unless explicitly stated otherwise.

Example 2: Evaluate $\displaystyle\oint_\Gamma \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz$, where Γ is the unit circle.

Solution: Let f be the function that appears as the integrand of the given integral. It is clear that f is analytic on and inside Γ except for a singularity when $z = 0$. Using Theorem 1 in Section 6.1, we find that

$$\operatorname{Res}(f, 0) = \frac{d}{dz}\Big(\frac{e^{iz}}{(z-2)(z+5i)}\Big)\Big|_{z=0} = \frac{(-10i)(i) - (1)(-2+5i)}{(-10i)^2} = \frac{12-5i}{-100}.$$

(We have combined the differentiate and evaluate steps, but carefully included the appropriate values.) It follows that

$$\oint_\Gamma \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz = 2\pi\,i\Big(\frac{12-5i}{-100}\Big) = \pi\Big(\frac{12i+5}{-50}\Big) = -\frac{\pi}{10} - \frac{6\pi}{25}\,i.$$

Example 3: Evaluate $\displaystyle\oint_\Gamma \frac{3z+2}{z^4+1}\,dz$, where Γ is the rectangle with vertices -2, 4, $4+3i$, and $-2+3i$.

Solution: Let f be the function that appears as the integrand of the given integral. The singularities of f occur at the four fourth roots of -1. These are

$$\begin{aligned} z_1 &= e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i); \\ z_2 &= e^{3i\pi/4} = \frac{1}{\sqrt{2}}(-1+i); \\ z_3 &= e^{5\pi/4} = \frac{1}{\sqrt{2}}(-1-i); \\ z_4 &= e^{7i\pi/4} = \frac{1}{\sqrt{2}}(1-i); \end{aligned}$$

which satisfy

$$\begin{aligned} z_1^2 = i = z_3^2, &\quad z_3 = -z_1; \\ z_2^2 = -i = z_4^2, &\quad z_4 = -z_2; \\ z_1 z_2 &= -1 = z_3 z_4; \\ z_1 + z_2 &= i\sqrt{2} = -(z_3 + z_4); \\ z_1 - z_2 &= \sqrt{2} = -(z_3 - z_4). \end{aligned}$$

We may not use all of these facts, but we record them for the sake of completeness. In addition, note that only z_1 and z_2 lie within Γ so we do not need the residues at the other two points. However, we determine all of the residues for further practice. Here are several options for finding the value of the integral.

Using the result in Example 2 of Section 6.1, we find that

$$\begin{aligned}
\operatorname{Res}(f, z_1) &= \frac{3z_1+2}{4z_1^3} = \frac{3z_1+2}{4z_1\cdot z_1^2} = \frac{1}{4i}\left(3+\frac{2}{z_1}\right) = \frac{1}{4i}(3-2z_2);\\
\operatorname{Res}(f, z_2) &= \frac{3z_2+2}{4z_2^3} = \frac{3z_2+2}{4z_2\cdot z_2^2} = \frac{-1}{4i}\left(3+\frac{2}{z_2}\right) = \frac{1}{4i}(-3+2z_1);\\
\operatorname{Res}(f, z_3) &= \frac{3z_3+2}{4z_3^3} = \frac{3z_3+2}{4z_3\cdot z_3^2} = \frac{1}{4i}\left(3+\frac{2}{z_3}\right) = \frac{1}{4i}(3+2z_2);\\
\operatorname{Res}(f, z_4) &= \frac{3z_4+2}{4z_4^3} = \frac{3z_4+2}{4z_4\cdot z_4^2} = \frac{-1}{4i}\left(3+\frac{2}{z_4}\right) = \frac{1}{4i}(-3-2z_1).
\end{aligned}$$

Notice that the sum of all four residues is 0. Hence, the value of the integral in Exercise 6.1.3d is 0. For our problem, we find that

$$\oint_\Gamma \frac{3z+2}{z^4+1}\,dz = 2\pi i\big(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\big) = \frac{2\pi i}{4i}(-2z_2+2z_1) = \pi(z_1-z_2) = \pi\sqrt{2}.$$

The answer is the same for any contour that contains z_1 and z_2 and does not contain z_3 and z_4. Suppose a contour contains just z_1 and z_3. (You should sketch a few appropriate possibilities for such contours.) In that case, the value of the integral is 3π.

To illustrate another way to find these residues, note that

$$f(z) = \frac{3z+2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}.$$

It then follows that

$$\begin{aligned}
\operatorname{Res}(f, z_1) &= \left.\frac{3z+2}{(z-z_2)(z-z_3)(z-z_4)}\right|_{z=z_1}\\
&= \frac{3z_1+2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}\\
&= \frac{\frac{3}{\sqrt{2}}(1+i)+2}{\sqrt{2}\cdot\sqrt{2}(1+i)\cdot\sqrt{2}\,i}\\
&= \frac{3+2\sqrt{2}+3i}{4(-1+i)}\\
&= -\frac{1}{4}\cdot\frac{3+2\sqrt{2}+3i}{1-i}\cdot\frac{1+i}{1+i}\\
&= -\frac{1}{4}\cdot\frac{2\sqrt{2}+(6+2\sqrt{2}\,)i}{2}\\
&= -\frac{1}{4}\big(\sqrt{2}+(3+\sqrt{2}\,)i\big)\\
&= \frac{1}{4i}\big(3+\sqrt{2}(1-i)\big)\\
&= \frac{1}{4i}(3-2z_2).
\end{aligned}$$

The last two steps are not necessary, but they do show that the two calculations give the same answer. You can certainly see how the first method has some clear advantages in terms of reducing the number of computations.

Suppose that a and b are real numbers that satisfy $0 < a < b$ and consider the four integrals

$$\int_0^{2\pi} \frac{1}{b + a\sin\theta}\, d\theta, \quad \int_0^{2\pi} \frac{1}{b - a\sin\theta}\, d\theta, \quad \int_0^{2\pi} \frac{1}{b + a\cos\theta}\, d\theta, \quad \int_0^{2\pi} \frac{1}{b - a\cos\theta}\, d\theta.$$

We claim that all four of these integrals are equal. Given that the functions are periodic and the integral is over one full period, this is not all that surprising. To prove these facts more carefully is a bit tedious, but here goes. We first note that the substitutions $\phi = \theta + \pi$ and $\phi = \theta - \pi$ yield

$$\int_0^{\pi} \frac{1}{b + a\sin\theta}\, d\theta = \int_{\pi}^{2\pi} \frac{1}{b + a\sin(\phi - \pi)}\, d\phi = \int_{\pi}^{2\pi} \frac{1}{b - a\sin\phi}\, d\phi;$$
$$\int_{\pi}^{2\pi} \frac{1}{b + a\sin\theta}\, d\theta = \int_0^{\pi} \frac{1}{b + a\sin(\phi + \pi)}\, d\phi = \int_0^{\pi} \frac{1}{b - a\sin\phi}\, d\phi;$$

respectively. Adding these two integral equalities shows that the first two integrals listed above are equal. We can prove that the third and fourth integrals are equal in the same way. To prove that the first and fourth integrals are equal, we first make the substitution $\phi = \theta + \frac{1}{2}\pi$ to obtain

$$\int_0^{2\pi} \frac{1}{b + a\sin\theta}\, d\theta = \int_{\pi/2}^{5\pi/2} \frac{1}{b + a\sin(\phi - \frac{1}{2}\pi)}\, d\phi = \int_{\pi/2}^{5\pi/2} \frac{1}{b - a\cos\phi}\, d\phi$$

and then note that

$$\int_{2\pi}^{5\pi/2} \frac{1}{b - a\cos\phi}\, d\phi = \int_0^{\pi/2} \frac{1}{b - a\cos\phi}\, d\phi$$

due to the period of cosine. The equality of the second and third integrals is verified in the same way.

As a result of the equality of these four integrals, the definite integral that appears as Exercise 6.2.5 gives the value of many related integrals.

The key result for Section 6.3 can be put in the following form. It is rather intriguing that integrals involving real-valued functions can be evaluated using complex numbers.

Theorem 4: Let P and Q be polynomials with real coefficients such that $Q(x) \neq 0$ for all x and the degree of Q is at least two more than the degree of P. If the zeros of Q in the upper half plane are $z_1, z_2, \ldots, z_n$, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(P/Q, z_k).$$

Here are a few examples of integrals that could be evaluated using Theorem 4:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx, \qquad \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\, dx, \qquad \int_{-\infty}^{\infty} \frac{x^4}{x^6+1}\, dx, \qquad \int_0^{\infty} \frac{x^4 - 3x^2 + 5}{x^8 + 12}\, dx.$$

Note that the last integral has a different lower bound. However, since the function is even, we can integrate over the entire real line and then divide by two. You might note that the second integral can be evaluated using simple calculus techniques.

Referring to functions $f:\mathbb{R}\to\mathbb{R}$, recall that f is an even function if $f(-x)=f(x)$ for all x and that f is an odd function if $f(-x)=-f(x)$ for all x. It is then easy to show that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{and} \qquad \int_{-a}^{a} f(x)\,dx = 0$$

for even and odd functions, respectively. For example, we see that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\int_{0}^{\infty} \frac{x^2}{x^4+1}\,dx, \qquad \int_{-\infty}^{\infty} \frac{x^3}{x^8+12}\,dx = 0, \qquad \int_{-\infty}^{\infty} \frac{x\cos x}{(x^2+1)^2}\,dx = 0.$$

It is helpful be keep these facts in mind when doing integrals of these types.

Example 5: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{3}{(x^2+9)^2}\,dx$.

Solution: Consider the rational function $R(z)=\dfrac{3}{(z^2+9)^2}$. The only zero of the denominator in the upper half plane is $3i$. Noting that

$$\operatorname{Res}(R,3i) = \frac{d}{dz}\frac{3}{(z+3i)^2}\Big|_{z=3i} = \frac{-6}{(6i)^3} = \frac{1}{36i},$$

we find that (using Theorem 4)

$$\int_{-\infty}^{\infty} \frac{3}{(x^2+9)^2}\,dx = 2\pi i\cdot\frac{1}{36i} = \frac{\pi}{18}.$$

You should use this result to verify the value of the integral in Exercise 6.3.5 by making an appropriate substitution and WITHOUT doing any actual integration.

Example 6: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$.

Solution: Consider the rational function $R(z)=\dfrac{1}{z^6+1}$. The zeros of the denominator in the upper half plane are $z_1=e^{i\pi/6}$, $z_2=e^{i\pi/2}$, and $z_3=e^{i5\pi/6}$. Using Theorem 4 and our quick way to find the residues of simple poles (Example 2 in Section 6.1), we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx &= 2\pi i\Big(\frac{1}{6z_1^5}+\frac{1}{6z_2^5}+\frac{1}{6z_3^5}\Big)\\
&= \frac{\pi i}{3}\Big(\frac{1}{z_1^3z_1^2}+\frac{1}{z_2^3z_2^2}+\frac{1}{z_3^3z_3^2}\Big)\\
&= \frac{\pi i}{3}\Big(\frac{1}{iz_1^2}-\frac{1}{iz_2^2}+\frac{1}{iz_3^2}\Big)\\
&= \frac{\pi}{3}\big(e^{-i\pi/3}-e^{-i\pi}+e^{-i5\pi/3}\big)\\
&= \frac{\pi}{3}\Big(\frac{1}{2}-\frac{\sqrt{3}}{2}i-(-1)+\frac{1}{2}+\frac{\sqrt{3}}{2}i\Big)\\
&= \frac{2\pi}{3}.
\end{aligned}$$

The steps involving the cubes of the roots are not necessary (you could go straight to the exponentials); they are included to point out simplifications/observations that are helpful in some cases.

The key result for Section 6.4 can be put in the following form.

Theorem 7: Let P and Q be polynomials with real coefficients such that $Q(x) \neq 0$ for all x and the degree of Q is at least one more than the degree of P. If the zeros of Q in the upper half plane are $z_1, z_2, \ldots, z_n$, then

$$\int_{-\infty}^{\infty} \frac{P(x)e^{imx}}{Q(x)}\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

where m is a positive number and f is the function defined by $f(z) = e^{imz}P(z)/Q(z)$.

Example 8: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4}\, dx$.

Solution: This is Example 1 in Section 6.4, but we will solve the problem in a simpler way. Note that

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4}\, dx = \int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4}\, dx + i\int_{-\infty}^{\infty} \frac{\sin(3x)}{x^2+4}\, dx = \int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\, dx$$

since $\sin(3x)$ is an odd function. Let $f(z) = e^{3iz}/(z^2+4)$ and note that the only 0 of the denominator that lies in the upper half plane is $2i$. Applying Theorem 7, we find that

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\, dx = 2\pi i \operatorname{Res}(f, 2i) = 2\pi i \cdot \left.\frac{e^{3iz}}{z+2i}\right|_{z=2i} = 2\pi i \cdot \frac{e^{-6}}{4i} = \frac{\pi}{2e^6}.$$

Hence, the value of the requested integral is $\pi/(2e^6)$.

Example 9: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)^2}\, dx$, where a is a positive number.

Solution: This time, we note that

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)^2}\, dx = \int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)^2}\, dx - i\int_{-\infty}^{\infty} \frac{x\cos x}{(x^2+a^2)^2}\, dx = \int_{-\infty}^{\infty} \frac{-ixe^{ix}}{(x^2+a^2)^2}\, dx$$

since $x\cos x$ is an odd function. Let $f(z) = -ize^{iz}/(z^2+a^2)^2$ and note that the only 0 of the denominator that lies in the upper half plane is ai. Applying Theorem 7, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)^2}\, dx &= \int_{-\infty}^{\infty} \frac{-ixe^{ix}}{(x^2+a^2)^2}\, dx \\
&= 2\pi i \operatorname{Res}(f, ai) \\
&= 2\pi \cdot \frac{d}{dz}\left.\frac{ze^{iz}}{(z+ai)^2}\right|_{z=ai} \\
&= 2\pi \cdot \frac{(2ai)^2\big(ai(ie^{-a}) + e^{-a}\big) - aie^{-a}2(2ai)}{(2ai)^4} \\
&= \frac{2\pi}{(2ai)^2e^a} \cdot (-a+1-1) \\
&= \frac{\pi}{2ae^a}.
\end{aligned}$$

Finding integral results involving a parameter such as this one can be helpful.

Theorem 10: (Rouché's Theorem) Suppose that f and g are analytic on and inside a simple closed contour C. If $|g(z)| < |f(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside C.

Proof: For each real number $t \in [0,1]$, let

$$N(t) = \frac{1}{2\pi i} \oint_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\, dz.$$

Since

$$|f(z) + tg(z)| \geq |f(z)| - |tg(z)| \geq |f(z)| - |g(z)| > 0$$

for all $t \in [0,1]$ and all $z \in C$, the Argument Principle (see Corollary 1 in Section 6.7) tells us that $N(t)$ is the number of zeros of $f(z) + tg(z)$ inside C. (Note that the hypotheses reveal that $f(z) + tg(z)$ does not have any poles inside C.) Since the function $N(t)$ is integer-valued and continuous, we must have $N(0) = N(1)$. It follows that f and $f+g$ have the same number of zeros inside C.

Example 11: Show that all of the roots of $p(z) = z^7 + 4z^5 - 3z^4 + 6z^2 - z + i$ lie within the circle $|z| = 3$.

Solution: Let $f(z) = z^7$ and $g(z) = 4z^5 - 3z^4 + 6z^2 - z + i$. Both f and g are entire functions and f has seven roots (counting multiplicities) within the circle $|z| = 3$. On the circle $|z| = 3$, we find that

$$\begin{aligned}
|g(z)| &\leq 4|z|^5 + 3|z|^4 + 6|z|^2 + |z| + |i| \\
&= 4 \cdot 3^5 + 3^5 + 6 \cdot 3^2 + 3 + 1 \\
&< 5 \cdot 3^5 + 3^5 \\
&< 3^7 \\
&= |f(z)|.
\end{aligned}$$

By Rouché's Theorem, the functions f and $p = f + g$ have the same number of zeros inside the circle $|z| = 3$. Hence, all seven roots of the polynomial p lie inside the circle $|z| = 3$.

Example 12: Show that $q(z) = z^4 - (2+i)z + 25$ has no roots inside the circle $|z| = 2$.

Solution: Let $f(z) = 25$ and $g(z) = z^4 - (2+i)z$. Both f and g are entire functions and f has no roots inside the circle $|z| = 2$. On the circle $|z| = 2$, we find that

$$|g(z)| \leq |z|^4 + |2+i|\,|z| = 16 + 2\sqrt{5} < 25 = |f(z)|.$$

By Rouché's Theorem, the functions f and $q = f + g$ have the same number of zeros inside the circle $|z| = 2$. Hence, the polynomial q has no roots inside the circle $|z| = 2$.

Recall that a domain is an open connected set. (You should review the terms 'open' and 'connected' if you do not remember them.) If f is analytic on a domain D, then the set $f(D) = \{f(z) : z \in D\}$ is the range of f. We sometimes refer to $f(D)$ as the image of D under the mapping f. For functions $f\colon \mathbb{C} \to \mathbb{C}$, both D and $f(D)$ are subsets of the complex plane. On the real line, the only connected sets are intervals. However, for functions $f\colon \mathbb{R} \to \mathbb{R}$ the image of an open interval may not be an open interval. For example, the function $f(x) = 1 - x^2$ maps the open interval $(-1, 1)$ to the half-open interval $(0, 1]$, and the function $g(x) = \sin x$ maps the open interval $(0, 8)$ to the closed interval $[-1, 1]$. This situation does not occur for complex valued functions

Theorem 13: (Open Mapping Theorem) If f is analytic and nonconstant in a domain D, then $f(D)$ is a domain.

Proof: To show that $f(D)$ is connected, let w_1 and w_2 be two points in $f(D)$. Since w_1 and w_2 are in the range of f, we can select points z_1 and z_2 in D for which $f(z_1) = w_1$ and $f(z_2) = w_2$. Since D is a connected set, there exists a path γ in D that joins the points z_1 and z_2. It then follows that $f \circ \gamma$ is a path in $f(D)$ that joins the points w_1 and w_2. (We are using the fact that the composition of two continuous functions is continuous and that $f(\gamma(t)) \in f(D)$ for all t values.) Hence, the set $f(D)$ is connected.

To show that $f(D)$ is open, we must show that all of its points are interior points. Let $w_0 \in f(D)$ and choose $z_0 \in D$ so that $f(z_0) = w_0$. Since the zeros of analytic functions are isolated (apply Corollary 3 in Section 5.6 to the function $f(z) - w_0$), there exists $\rho > 0$ such that $f(z) \neq w_0$ for all z that satisfy $0 < |z - z_0| \leq \rho$. Since D is an open set, we may also assume that ρ has been chosen small enough so that every z that satisfies $|z - z_0| \leq \rho$ belongs to D. Now let $\delta = \min\{|f(z) - w_0| : |z - z_0| = \rho\}$. Since each of the numbers $|f(z) - w_0|$ in this set is positive, it can be shown that the number δ is positive. Suppose that w_1 is a complex number that satisfies $|w_1 - w_0| < \delta$. Noting that

$$|w_0 - w_1| < \delta \leq |f(z) - w_0| \quad \text{for all } z \text{ that satisfy } |z - z_0| = \rho,$$

Rouché's Theorem tells us that the functions

$$f(z) - w_0 \qquad \text{and} \qquad (f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$$

have the same number of zeros inside the circle $|z - z_0| = \rho$. In particular, there exists at least one point $z_1 \in D$ such that $f(z_1) = w_1$. This shows that $w_1 \in f(D)$. Since w_1 was an arbitrary point in the set $|w - w_0| < \delta$, we see that there is a disk around w_0 that lies inside $f(D)$. This shows that w_0 is an interior point of $f(D)$. We conclude that $f(D)$ is an open set.