**Some notes for Chapter 7**

**Definition 1:** Let $A$ and $B$ be nonempty sets. A function $f\colon A \to B$ is one-to-one (or injective) if each $b \in B$ has at most one preimage in $A$.

There are several equivalent ways to state this definition:

$$\begin{aligned} f \text{ is one-to-one} \quad &\Leftrightarrow \quad \text{each } b \text{ in } B \text{ has at most one preimage in } A \\ &\Leftrightarrow \quad \text{for each } b \in B, \text{ the set } \{a \in A : f(a) = b\} \text{ contains at most one element} \\ &\Leftrightarrow \quad \text{for each } a_1 \text{ and } a_2 \text{ in } A \text{ if } f(a_1) = f(a_2), \text{ then } a_1 = a_2 \\ &\Leftrightarrow \quad \text{for each } a_1 \text{ and } a_2 \text{ in } A \text{ if } a_1 \neq a_2, \text{ then } f(a_1) \neq f(a_2). \end{aligned}$$

Sometimes one form of the definition is preferable over another. The following real-valued functions are one-to-one on the given interval:

$$f(x) = x^2 \text{ on } [0, \infty), \qquad g(x) = \sin(x/2) \text{ on } (0, \pi), \qquad h(x) = x^3 \text{ on } \mathbb{R}, \qquad F(x) = \frac{2x}{x-1} \text{ on } \mathbb{R} \setminus \{1\}.$$

You should be able to convince yourself that the functions $f$, $g$, and $h$ are one-to-one on the given intervals. For the function $F$, suppose that $F(s) = F(t)$ for real numbers $s$ and $t$ that do not equal 1. We then have

$$\frac{2s}{s-1} = \frac{2t}{t-1} \quad \Rightarrow \quad 2s(t-1) = 2t(s-1) \quad \Rightarrow \quad -2s = -2t \quad \Rightarrow \quad s = t.$$

Hence, the function $F$ is one-to-one on its domain (see the third form of the definition above). The following real-valued functions are not one-to-one on the given interval:

$$f(x) = x^2 \text{ on } (-2, 2), \qquad g(x) = \sin x \text{ on } (0, \pi), \qquad h(x) = x^2 + 2x - 3 \text{ on } \mathbb{R}, \qquad F(x) = \frac{x^2}{x^4+1} \text{ on } \mathbb{R}.$$

To verify this, we simply note that two different inputs generate the same output:

$$f(1) = f(-1), \qquad g(\pi/6) = g(5\pi/6), \qquad h(1) = h(-3), \qquad F(-2) = F(2).$$

(For later reference, note that $x^2$ is one-to-one on some domains but not on others.) For real-valued functions, there is a graphical interpretation of one-to-one: a function $f$ is one-to-one if each horizontal line intersects the graph of $y = f(x)$ in at most one point. One way to guarantee that this occurs involves the derivative. If $f'$ is positive (negative) on an interval $(a, b)$, then the function $f$ is strictly increasing (decreasing) on $(a, b)$ and thus one-to-one on $(a, b)$.

For complex-valued functions, there is no simple graphical interpretation for a function to be one-to-one; we just know that $f$ never maps two distinct points to the same image. In addition, functions that are one-to-one on $\mathbb{R}$ (such as $x^3$) may not be one-to-one on $\mathbb{C}$. An important example is the exponential function. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is one-to-one, but the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(x) = e^z$ is not one-to-one. For instance, we know that $e^0 = e^{2\pi i}$. On the other hand, recall that $e^z$ is one-to-one on the infinite strip $0 < \operatorname{Im} z < \pi$. We are often interested in local behavior such as this.

As mentioned above, there is a connection between the derivative of a real-valued function and conditions that guarantee that the function is one-to-one. For complex-valued functions, we have the following result.

**Theorem 2:** If $f$ is analytic at $z_0$ and $f'(z_0) \neq 0$, then there exists $\rho > 0$ such that $f$ is one-to-one on the set $\{z : |z - z_0| < \rho\}$.

**Proof:** Recall that $f$ is analytic at $z_0$ means that $f$ is differentiable in some disk containing $z_0$. Consequently, there exists $r > 0$ such that $f$ is differentiable on the set $\{z : |z - z_0| < r\}$. We also know that $f'$ is continuous on this set. By the definition of continuity, for the positive number $|f'(z_0)|/2$, there exists a positive number $\rho$ such that $\rho < r$ and

$$|f'(z) - f'(z_0)| < \frac{|f'(z_0|}{2}$$

for all $z$ that satisfy $|z - z_0| < \rho$. Let $D = \{z : |z - z_0| < \rho\}$. We claim that $f$ is one-to-one on $D$.

Suppose that $z_1$ and $z_2$ are distinct points in $D$ and let $\gamma$ be the line segment in $D$ joining these two points (from $z_1$ to $z_2$). We first note that

$$\int_\gamma f'(z_0)\,dz = f'(z_0)(z_2 - z_1)$$

and (using what we have called the ML Lemma: Theorem 5 on page 170)

$$\Big|\int_\gamma \big(f'(z_0) - f'(z)\big)\,dz\Big| \leq \int_\gamma \big|f'(z_0) - f'(z)\big|\,dz \leq \frac{|f'(z_0|}{2}\,|z_2 - z_1|.$$

It follows that

$$\begin{aligned}
|f(z_2) - f(z_1)| &= \Big|\int_\gamma f'(z)\,dz\Big| \\
&= \Big|\int_\gamma f'(z_0)\,dz - \int_\gamma \big(f'(z_0) - f(z)\big)\,dz\Big| \\
&\geq \Big|\int_\gamma f'(z_0)\,dz\Big| - \Big|\int_\gamma \big(f'(z_0) - f(z)\big)\,dz\Big| \\
&= |f'(z_0)|\,|z_2 - z_1| - \Big|\int_\gamma \big(f'(z_0) - f(z)\big)\,dz\Big| \\
&\geq |f'(z_0)|\,|z_2 - z_1| - \frac{|f'(z_0|}{2}\,|z_2 - z_1| \\
&= \frac{|f'(z_0|}{2}\,|z_2 - z_1|.
\end{aligned}$$

Since this last number is not zero, we find that $f(z_1) \neq f(z_2)$. Hence, the function $f$ is one-to-one on the set $D$.

In calculus, we consider tangent lines and note that $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$, where the approximation is very good for values of $x$ near $x_0$. In other words, the function $f$ behaves very much like a linear function near the point $x_0$. (You may recall that this was the basis for Newton's method for approximating roots.) For complex variables, we write $f(z) \approx f'(z_0)(z - z_0) + f(z_0)$, but there is no tangent line analogy. However, the function $f$ does behave like this linear function. In particular, the mapping $f$ at $z_0$ rotates lines through an angle $\arg f'(z_0)$. (We are assuming that $f'(z_0) \neq 0$.) Since $f$ does the same thing to all lines (including tangent lines) through $z_0$, the angles between contours in the $z$-plane (the domain) are unchanged in the $w$-plane (the images). A mapping that preserves angles is said to be conformal. You may refer to the book to see a graphical example of the preservation of angles under a mapping as well as read a proof of the following theorem.

**Theorem 3:** An analytic function $f$ is conformal at every point $z_0$ for which $f'(z_0) \neq 0$.

Hence, analytic functions map domains to domains and they preserve the angles made by the boundaries of the domains. The following example (it is a portion of Exercise 7.2.13b) illustrates these ideas.

**Example 4:** Determine the image of the half-strip $\{z : 0 < \operatorname{Re} z < \frac{1}{2}\pi, \operatorname{Im} z > 0\}$ under the mapping $\cos z$.

**Solution:** With $z = x + iy$, we know that $\cos z = \cos x \cosh y - i \sin x \sinh y$ (see Exercise 3.2.13). Hence, we are concerned with the mapping

$$u(x,y) = \cos x \cosh y; \quad v(x,y) = -\sin x \sinh y.$$

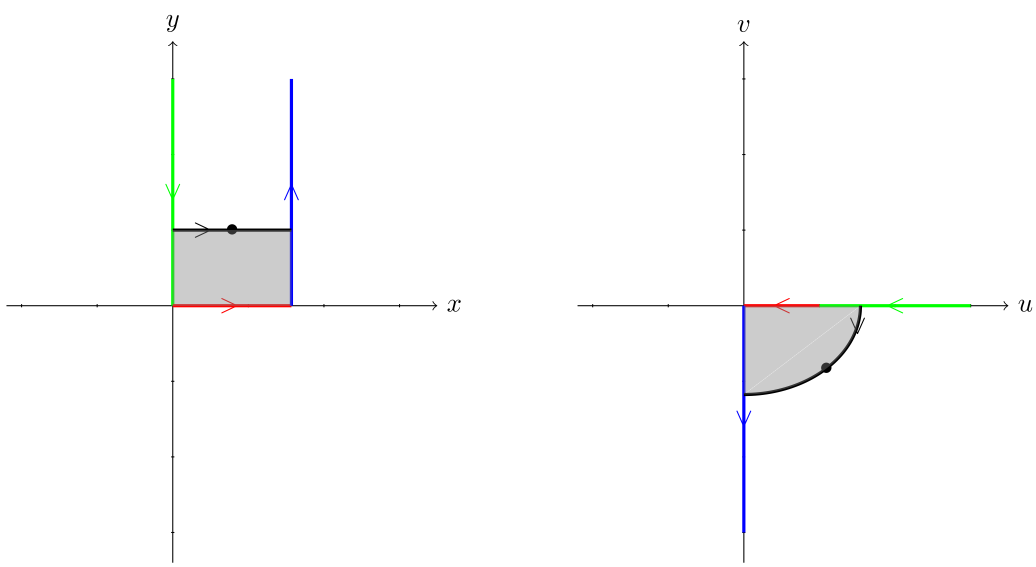
Looking at the three boundaries (and referring to the graphs), we find that

(green) for $x = 0$, $0 \leq y < \infty$, we have $u = \cosh y$, $v = 0$ so $1 \leq u < \infty$;

(red) for $0 \leq x \leq \pi/2$, $y = 0$, we have $u = \cos x$, $v = 0$ so $0 \leq u \leq 1$;

(blue) for $x = \pi/2$, $0 \leq y < \infty$, we have $u = 0$, $v = -\sinh y$ so $-\infty < v \leq 0$.

Note that the arrows indicate the corresponding directions of travel.




Since domains map to domains, the image is either the fourth quadrant or the entire combination of quadrants one, two, and three. There are two ways to determine the correct domain for the image. The first is follow the contour in the direction of the arrows; the region to the left of the contour in the $z$-plane corresponds to the region to the left of the contour in the $w$-plane. Hence, the image is the fourth quadrant (not including the boundaries). The other method is to pick a sample point. The bullet point in the $z$-plane is the point $\frac{1}{4}\pi + i$ and its image (the bullet point in the $w$-plane) is

$$\frac{\cosh 1}{\sqrt{2}} - \frac{\sinh 1}{\sqrt{2}}\, i.$$

Since this point is interior to the given domain, its image must be in the interior of the image domain. Hence, the image of the half-strip is the fourth quadrant.

There are some other observations that should be made concerning the figure. First of all, note that the red and blue curves (lines in this case) meet at right angles in both the $z$-plane and the $w$-plane. This illustrates the fact that the mapping is conformal. (Do you see why the green and red images in the $w$-plane do not meet at right angles along the $x$-axis?) Next, the black line in the $z$-plane is given by $z = i$ for $0 < \operatorname{Re} z < \frac{1}{2}\pi$. Its image under the mapping $\cos z$ is

$$u(x, y) = \cos x \cosh 1; \quad v(x, y) = -\sin x \sinh 1 \qquad \Leftrightarrow \qquad \frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1.$$

The image of this line is thus a portion of an ellipse in the $w$-plane. The image of the shaded rectangle in the $z$-plane is the shaded portion of the ellipse in the $w$-plane. Note that the angles where the black, green, and blue curves meet are still preserved (right angles in the figure).

The set of all Möbius transformations (sometimes called linear fractional transformations or LFTs) is an important collection of one-to-one conformal mappings. A Möbius transformation $f$ has the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a$, $b$, $c$ and $d$ are complex constants that satisfy $ad - bc \neq 0$. This last condition guarantees that $f'(z)$ is never 0. For $c \neq 0$, we note that

$$\frac{az + b}{cz + d} = \frac{1}{c} \cdot \frac{az + \frac{ad}{c} - \frac{ad}{c} + b}{z + (d/c)} = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + (d/c)}.$$

This shows that a Möbius transformation can be decomposed into the simpler operations of translation, magnification, rotation, and inversion. You should refer to the textbook for information on these simpler mappings. As a result, Möbius transformations map the collection of lines and circles to itself; see Theorem 5 in Section 7.3.

We can sometimes find appropriate Möbius transformations by inspection.

**Example 5:** Find a Möbius transformation that maps the points 1, 2, $\infty$ to $\infty$, 0, $i$, respectively.

**Solution:** We know that the pole occurs at 1, that a 0 occurs at 2, and that the limit as $|z|$ goes to infinity must be $i$. It follows that the desired function $f$ is

$$f(z) = i\left(\frac{z - 2}{z - 1}\right) = \frac{iz - 2i}{z - 1}.$$

**Example 6:** Find a Möbius transformation that maps the points 0, 1, $\infty$ to 2, $\infty$, $i$, respectively.

**Solution:** We know that the pole occurs at 1 and that the limit as $|z|$ goes to infinity must be $i$. It follows that the desired function $g$ has the form

$$g(z) = i\left(\frac{z + b}{z - 1}\right).$$

In order for $g(0) = 2$, we must have $-bi = 2$. It follows that $b = 2i$ and thus

$$g(z) = i\left(\frac{z + 2i}{z - 1}\right) = \frac{iz - 2}{z - 1}.$$

It is always a good idea to double check your final answer.

It takes three distinct points to determine a unique Möbius transformation (see Exercise 7.4.8). The previous two examples illustrate this fact. However, we need a more structured approach to find these mappings. One simple type of LFT is given by the mapping that takes the points $z_1$, $z_2$, $z_3$ to the points 0, 1, $\infty$. Note that this mapping takes the directed line or circle determined by the three points $z_1$, $z_2$, $z_3$ to the real axis. Assuming that the three points are finite, it is easy to see that the appropriate function $f$ is given by

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \equiv (z, z_1, z_2, z_3);$$

the latter form of the function is called the cross-ratio for the four points $z$, $z_1$, $z_2$, $z_3$. (If one of the numbers is $\infty$, just cancel the two terms that involve $\infty$.) It is not difficult to remember this formula. Now suppose that we want a function $f$ that takes the points $z_1$, $z_2$, $z_3$ to the points $w_1$, $w_2$, $w_3$. Letting

$$T(z) = (z, z_1, z_2, z_3), \quad S(w) = (w, w_1, w_2, w_3), \qquad \text{we want} \qquad f(z) = S^{-1}(T(z)).$$

To see why this works, note that

$$f(z_1) = S^{-1}(T(z_1)) = S^{-1}(0) = w_1;$$
$$f(z_2) = S^{-1}(T(z_2)) = S^{-1}(1) = w_2;$$
$$f(z_3) = S^{-1}(T(z_3)) = S^{-1}(\infty) = w_3.$$

To find a formula for $f(z)$, we solve the equation

$$S(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = T(z)$$

for $w$ as a function of $z$. These calculations become extremely tedious even for simple cases.

**Example 7:** Find a Möbius transformation that takes the points $i$, 4, $1 + i$ to the points 1, 2, 3.

**Solution:** We must solve the equation

$$\frac{(w - 1)(-1)}{(w - 3)(1)} = \frac{(z - i)(3 - i)}{(z - 1 - i)(4 - i)}$$

for $w$. Omitting the details, we find that

$$w = \frac{(-56 + 3i)z + (20 + 56i)}{(-30 + i)z + (18 + 30i)}.$$

Calling this function $f$, we can check this function to see if it actually works:

$$f(i) = \frac{-56i - 3 + 20 + 56i}{-30i - 1 + 18 + 30i} = \frac{17}{17} = 1;$$
$$f(4) = \frac{-224 + 12i + 20 + 56i}{-120 + 4i + 18 + 30i} = \frac{-204 + 68i}{-102 + 34i} = 2;$$
$$f(1 + i) = \frac{-56 + 3i - 56i - 3 + 20 + 56i}{-30 + i - 30i - 1 + 18 + 30i} = \frac{-39 + 3i}{-13 + i} = 3.$$

Hence, we have found the requested Möbius transformation.

As an interesting observation, we point out a connection between Möbius transformations and $2 \times 2$ matrices. We associate the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with the mapping} \qquad f(z) = \frac{az+b}{cz+d}\,.$$

It then turns out that composition of functions is matrix multiplication (see the top of page 396 in the textbook and read off the appropriate values for the corresponding matrix) and the inverse of a function is essentially the inverse of the matrix. As a reminder, recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The "essentially" part means that we ignore the scalar factor, that is, the inverse of $f$ is the function

$$\frac{dz-b}{-cz+a} \qquad \text{associated with} \qquad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(see the bottom of page 395 in the textbook).

To illustrate these ideas with the previous example, the function $T$ mapping $i$, 4, $1+i$ to 0, 1, $\infty$ is

$$T(z) = \frac{(z-i)(3-i)}{(z-1-i)(4-i)} = \frac{(3-i)z-(1+3i)}{(4-i)z-(5+3i)} \qquad \text{or} \qquad \begin{pmatrix} 3-i & -(1+3i) \\ 4-i & -(5+3i) \end{pmatrix}$$

and the function $S$ mapping 1, 2, 3 to 0, 1, $\infty$ is

$$S(z) = \frac{(z-1)(-1)}{(z-3)(1)} = \frac{-z+1}{z-3} \qquad \text{or} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$$

To find the function $f(z) = S^{-1}(T(z))$, we compute the matrix product

$$\begin{pmatrix} -3 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3-i & -(1+3i) \\ 4-i & -(5+3i) \end{pmatrix} = \begin{pmatrix} -13+4i & 8+12i \\ -7+2i & 6+6i \end{pmatrix}.$$

This answer looks different than the one we obtained as the solution in Example 7. However, each entry in the matrix can be multiplied by the same constant without changing the function (the constant cancels). In this case, we find that (using scalar multiplication)

$$(4+i)\begin{pmatrix} -13+4i & 8+12i \\ -7+2i & 6+6i \end{pmatrix} = \begin{pmatrix} -56+3i & 20+56i \\ -30+i & 18+30i \end{pmatrix}.$$

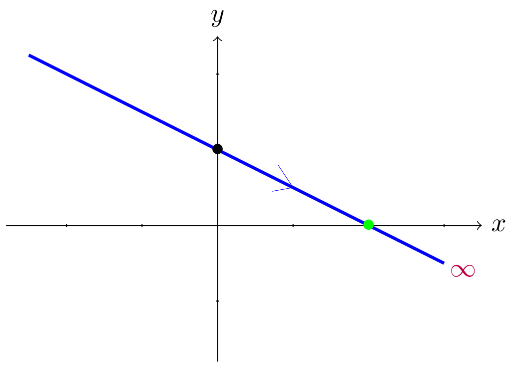
Hence, another representation for the function $f$ that maps the points $i$, 4, $1+i$ to the points 1, 2, 3 is

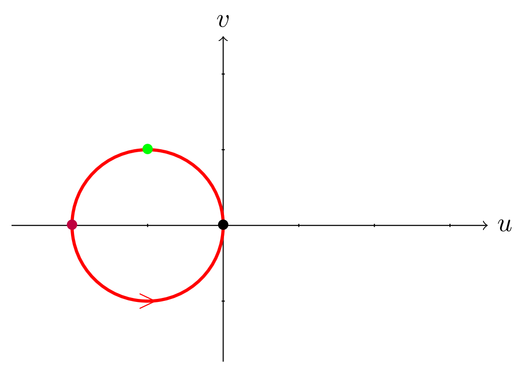
$$f(z) = \frac{(13-4i)z-(8+12i)}{(7-2i)z-(6+6i)}.$$

Most of the problems that we will consider are much less complicated.

**Example 8:** Find a Möbius transformation that maps the half-plane region above the line $x + 2y = 2$ to the interior of the circle $|z + 1| = 1$.

**Solution:** The two regions are indicated in the graphs given below.




As indicated by the color of the bullet points on each of the boundary curves, we want to find a Möbius transformation that takes the points $i$, 2, $\infty$ to the points 0, $-1+i$, $-2$. The arrows on the graphs indicate the corresponding directions of travel. Note that the left region for the blue line is the half-plane region above the line $x + 2y = 2$ and that the left region for the red circle is the interior of the circle $|z + 1| = 1$. Hence, our mapping will have the desired properties.

We offer two methods for the finding the function. For the first method, we use the fact that $f(i) = 0$ and $f(\infty) = -2$ to find that our function has the form $f(z) = \dfrac{-2(z-i)}{z+d}$ for some constant $d$. We then use the fact that $f(2) = -1 + i$ to find $d$:

$$\frac{-2(2-i)}{2+d} = -1+i \quad \Leftrightarrow \quad \frac{2(2-i)}{1-i} = 2+d \quad \Leftrightarrow \quad (2-i)(1+i) = 2+d \quad \Leftrightarrow \quad d = 1+i.$$

Hence, the requested Möbius transformation is $f(z) = \dfrac{-2z+2i}{z+1+i}$.

For a second method, we use the cross-ratio method to find the mappings $T$ taking $i$, 2, $\infty$ to 0, 1, $\infty$ and $S$ taking the points 0, $-1+i$, $-2$ to 0, 1, $\infty$. We thus have

$$T(z) = \frac{z-i}{2-i} \sim \begin{pmatrix} 1 & -i \\ 0 & 2-i \end{pmatrix} \quad \text{and} \quad S(z) = \frac{z(1+i)}{(z+2)(-1+i)} = \frac{-z}{z+2} \cdot \frac{1+i}{1-i} = \frac{-iz}{z+2} \sim \begin{pmatrix} -i & 0 \\ 1 & 2 \end{pmatrix}$$

The mapping that we want is $S^{-1} \circ T$ and this can be found using matrix multiplication:

$$\begin{pmatrix} 2 & 0 \\ -1 & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 2-i \end{pmatrix} = \begin{pmatrix} 2 & -2i \\ -1 & -1-i \end{pmatrix} \sim \frac{2z-2i}{-z-1-i} = \frac{-2z+2i}{z+1+i}.$$

This, of course, is the same answer we found using the other method.

As has been mentioned, we can always check that our function has the desired properties:

$$f(i) = 0, \quad f(2) = \frac{-4+2i}{3+i} = \frac{(-4+2i)(3-i)}{10} = \frac{-10+10i}{10} = -1+i, \quad f(\infty) = -2, \quad f(0) = \frac{2i}{1+i} = 1+i.$$

The last value shows that a point under the line maps to a point outside of the circle. Finally, the computation

$$\frac{-2z+2i}{z+1+i} = -1 \quad \Leftrightarrow \quad -2z+2i = -z-1-i \quad \Leftrightarrow \quad z = 1+3i$$

shows that the point $1 + 3i$ maps to the center of the circle.

**Example 9:** Find the image of the sector $-\pi/4 < \text{Arg}\, z < \pi/4$ under the mapping $f(z) = z/(z-1)$.

**Solution:** (This is Exercise 7.3.9 in the textbook.) Referring to the arrows on the left graph in the figure below and identifying appropriate points along the boundary, we note that

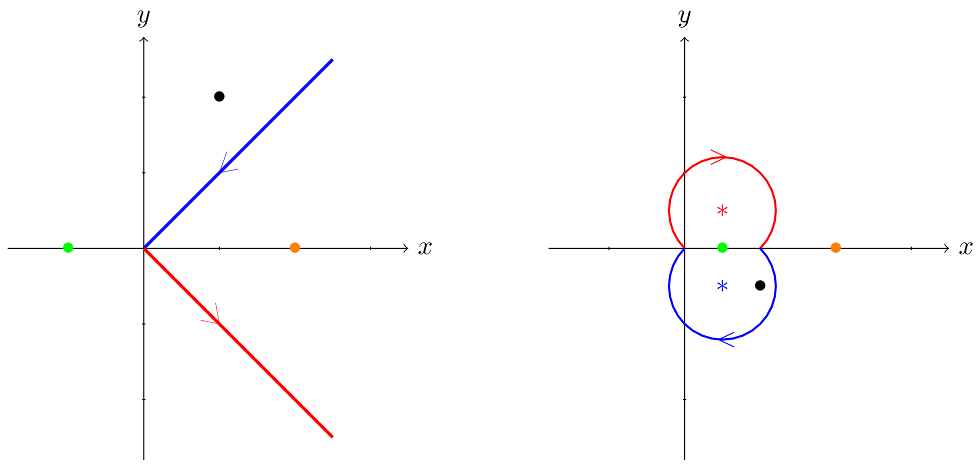
$$f(\infty) = 1, \quad f(1+i) = 1-i, \quad f(0) = 0, \quad f(1-i) = 1+i, \quad f(\infty) = 1.$$

Since the two lines do not go through the pole of the mapping, we know that the images of the lines will be circles. Given three points on a circle, we can find the center of the circle by finding the intersection of two perpendicular bisectors. (You should draw a picture to make sure you understand this.) For the blue [lower] image, we find that

the perpendicular bisector of the line segment from 1 to $1-i$ is $y = -\frac{1}{2}$;

the perpendicular bisector of the line segment from 1 to 0 is $x = \frac{1}{2}$.

It follows that the center of the circle is $\frac{1}{2} - \frac{1}{2}\,i$ and the radius is $\sqrt{2}/2$. (To find the radius, we just find the distance from the center to one point on the circle; the origin is the easiest in this case.)




Similarly, the center of the red [upper] circle is $\frac{1}{2} + \frac{1}{2}\,i$ and the radius is $\sqrt{2}/2$; the centers of the circles are marked with asterisks. (For the record, finding the center is not always this simple; the lines parallel to the coordinate axes make things easy here.) Noting which regions are on the left of the direction of travel, we find that the image of the sector $-\pi/4 < \text{Arg}\, z < \pi/4$ under the mapping $f(z) = z/(z-1)$ is the region that lies outside of the circles.

As indicated in the graphs, we can also map some specific points and see where they go. In this case, the color coordinated points and their images are determined from the following values of the function:

$$f(1+2i) = 1 - \frac{1}{2}\,i, \quad f(-1) = \frac{1}{2}, \quad f(2) = 2.$$

Once again, we see that the region exterior to the sector maps into the interior of the circles. (This corresponds to walking the paths and locating the region to the right.) As a final comment, note that the circles meet at right angles, that is, their corresponding tangent lines are perpendicular at the points of intersection. This is to be expected since the lines forming the boundary of the sector meet at right angles and the mapping is conformal.