1. Suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in a vector space V. Determine whether or not the set $\{\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}, \mathbf{v}-\mathbf{u}\}$ is linearly independent.

I Since $(u+w) - (v+w) + (v-u) = 0$, a nontrivial linear combination of the vectors gives the zero vector. Therefore, the set $\{u+w, v+w, v-u\}$ is linearly dependent.

II alternate option Suppose that $a(u+w) + b(v+w) + c(v-u) = 0$. Then $(a-c)\bar{u} + (b+c)v + (a+b)w = 0$. Since $\{u, v, w\}$ is linearly independent, we must have $a-c = b+c = a+b = 0$. Solving this system using a coefficient matrix yields

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that there are nontrivial solutions for a, b, and c. Hence, the set $\{u+w, v+w, v-u\}$ is linearly dependent.

2. Define $T : \cancel{M_{22}}\ M_{2x2} \to \mathcal{P}_1$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)t + b$. Find a basis for the kernel of T.

We need to find all matrices that map to the zero polynomial. So $a+d = 0$ and $b = 0$ must occur:

$T\left(\begin{bmatrix} a & 0 \\ c & -a \end{bmatrix}\right) = 0$ gives all solutions.

Since $\begin{bmatrix} a & 0 \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

a basis for the kernel of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

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just to illustrate how T works: $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 5t + 2$

3. Find all values of a and b for which the given system is consistent. Your answer should be in the form of an equation.

$$x_1 + 2x_2 + 11x_3 = a$$
$$x_1 - 4x_2 + 5x_3 = 7$$
$$x_2 + x_3 = b$$

We start by writing the augmented matrix and reducing it to echelon form

$$\begin{pmatrix} 1 & 2 & 11 & a \\ 1 & -4 & 5 & 7 \\ 0 & 1 & 1 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 11 & a \\ 0 & -6 & -6 & 7-a \\ 0 & 1 & 1 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 11 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & 6b+7-a \end{pmatrix}$$

We must have $6b+7-a=0$ to have solutions.

Therefore, the system is consistent for all values of a and b that satisfy $a-6b=7$.

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Find the standard matrix representing the transformation T.

The standard matrix for T has the form $[T(e_1) \;\; T(e_2)]$.

It is easy to see that

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since T is linear, it follows that

$$T(e_1) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and}$$

$$T(e_2) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

The standard matrix representing T is $\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}$.