Total: Name:

Math 126 **First Exam** **Spring 2011**

Write neat, concise, and accurate solutions to each of the problems below—I will not give partial credit for steps I cannot follow. Include all relevant steps, use correct notation, give sufficient details, and finish problems with a complete sentence when appropriate. Each problem is worth 6 points. No calculators are allowed for this exam.

1. State the definition of the integral. Be certain to include all of the appropriate words and symbols.

The integral of a continuous function f on an interval $[a, b]$, denoted $\int_a^b f(x)\,dx$, is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \frac{b-a}{n}.$$

2. For the function f defined by $f(x) = \int_2^{3x} \tan(t^2)\,dt$, find $f'(x)$.

Using the FTC and the chain rule,

$$f'(x) = 3\tan(9x^2)$$

59	81, 81
54	79
50	74
34, 34	70
21	65
53	74
45	67
37	60
30	53

3. Evaluate $\int \sin(2x)\cos^3(2x)\,dx$.

guess and check (mentally)

$$\int \sin(2x)\cos^3(2x)\,dx = -\frac{1}{8}\cos^4(2x) + C$$

could use integration by parts

$u = 2x$ $\quad dv = 2(2x+1)^{1/2}\,dx$

$du = 2dx$ $\quad v = \frac{2}{3}(2x+1)^{3/2}$

$$\frac{4}{3}x(2x+1)^{3/2} - \frac{4}{15}(2x+1)^{5/2} + C$$

4. Evaluate $\int 4x\sqrt{2x+1}\,dx$.

let $u = 2x+1$

then $du = 2dx$

and $4x = 2(u-1)$

$$\begin{aligned}\int 4x\sqrt{2x+1}\,dx &= \int 2(u-1)\,u^{1/2}\,\frac{1}{2}\,du\\ &= \int \left(u^{3/2} - u^{1/2}\right) du\\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C\\ &= \frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2} + C\end{aligned}$$

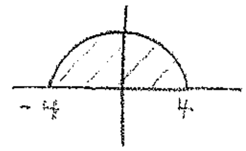
could factor

$$\frac{2}{15}u^{3/2}(3u-5)$$

$$\frac{4}{15}(3x-1)(2x+1)^{3/2}$$

5. Evaluate $\int_{-4}^{4} 5\sqrt{16-x^2}\,dx$.

represents area



$\frac{1}{2}\cdot\pi\cdot 4^2$

$$\int_{-4}^{4} 5\sqrt{16-x^2}\,dx = 5\int_{-4}^{4}\sqrt{16-x^2}\,dx$$

$$= 5\cdot 8\pi$$

$$= 40\pi$$

6. Evaluate $\int_{0}^{1} \frac{x^2+2}{x^3+6x+1}\,dx$.

$$\int_0^1 \frac{x^2+2}{x^3+6x+1}\,dx = \frac{1}{3}\ln\left|x^3+6x+1\right|\Big|_0^1$$

$$= \frac{1}{3}\ln 8$$

$$= \ln 2$$

7. Evaluate $\int_1^\infty \frac{x}{(x^2+4)^3}\,dx$.

$$\int_1^\infty \frac{x}{(x^2+4)^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{(x^2+4)^3}\,dx$$

$$= \lim_{b\to\infty} \left.\frac{-1}{4(x^2+4)^2}\right|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{-1}{4(b^2+4)^2} + \frac{1}{100}\right)$$

$$= \frac{1}{100}$$

8. Evaluate $\int 4xe^{x/2}\,dx$.

use integration by parts

$u = 4x \qquad dv = e^{x/2}\,dx$

$du = 4dx \qquad v = 2e^{x/2}$

$$\int 4xe^{x/2}\,dx = 8xe^{x/2} - \int 8e^{x/2}\,dx$$

$$= 8xe^{x/2} - 16e^{x/2} + C$$

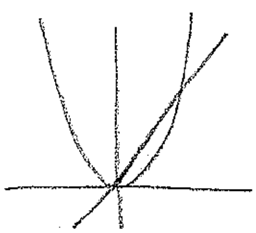
9. Suppose that $v(t) = t^3 - 4t$ gives the velocity in meters per second of a particle at time t seconds. Find the distance traveled by the particle during the time interval $0 \le t \le 3$.

$$t^3 - 4t = 0$$
$$t(t^2-4) = 0$$
$$t = -2, 0, 2$$

$$d = \int_0^3 |t^3 - 4t|\,dt$$
$$= \int_0^2 (4t - t^3)\,dt + \int_2^3 (t^3 - 4t)\,dt$$
$$= \left(2t^2 - \tfrac{1}{4}t^4\right)\Big|_0^2 + \left(\tfrac{1}{4}t^4 - 2t^2\right)\Big|_2^3$$
$$= (8-4) + \left(\tfrac{81}{4} - 18\right) - (4-8)$$
$$= \frac{81}{4} - 10$$
$$= \frac{41}{4}$$

The particle travels $\frac{41}{4}$ meters during this time period.

10. Find the area of the region bounded by the graphs of the equations $y = x^4$ and $y = 8x$.



$$x^4 = 8x$$
$$x^4 - 8x = 0$$
$$x(x^3 - 8) = 0$$
$$x = 0, 2$$

$$A = \int_0^2 (8x - x^4)\,dx$$
$$= \left(4x^2 - \tfrac{1}{5}x^5\right)\Big|_0^2$$
$$= 16 - \frac{32}{5}$$
$$= \frac{48}{5}$$

The area bounded by the curves is 9.6 square units.

omitted last 2 problems

estimate $\int_0^3 \sqrt[3]{8x^6+2}\ dx$

$\int_0^3 2x^2\,dx = 18$ vs 18.8075

evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(1+\frac{3i}{n}\right)^2$

$\int_1^4 x^2\,dx = 21$

additional problems

1. The derivative of a function f is another function f' defined by $f'(x) = \lim_{v\to x} \frac{f(v)-f(x)}{v-x}$ for all values of x in the domain of f for which the limit exists.
2. If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$, then $F'(x) = f(x)$ for all $x \in [a,b]$.
3. Since $\frac{1}{\sqrt{x^4-1}} > \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$ for all $x \in [2,4]$, it follows that

$$\int_2^4 \frac{1}{\sqrt{x^4-1}}\,dx > \int_2^4 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_2^4 = \frac{1}{4}.$$

4. a) $\int_4^{10} \sqrt{x^8+3}\,dx > \int_4^{10} x^4\,dx$ since $\sqrt{x^8+3} > \sqrt{x^8} = x^4$ on $[4,10]$.

   b) $\int_0^1 \cos^3 x\,dx > \int_0^1 \cos^5 x\,dx$ since $0 \le \cos^2 x \le 1$ and $\cos^3 x \ge 0$ on $[0,1]$ together imply that $\cos^5 x \le \cos^3 x$ on $[0,1]$.

Total: Name:

Math 126 **Second Exam** **Spring 2011**

Write neat, concise, and accurate solutions to each of the problems below—I will not give partial credit for steps I cannot follow. Include all relevant details, use correct notation, provide sufficient explanations, and finish problems with a complete sentence when appropriate. Each problem is worth 5 points. Calculators are not allowed for this exam.

1. State the definition of the integral (including all of the words).

The integral of a continuous function f on an interval $[a,b]$, denoted $\int_a^b f(x)\,dx$, is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+i\cdot\frac{b-a}{n}\right)\frac{b-a}{n}$$

2. Evaluate $\displaystyle\int \frac{8x-15}{x^2+25}\,dx$. split up

$$\int \frac{8x-15}{x^2+25}\,dx = \int\left(\frac{8x}{x^2+25} - \frac{15}{x^2+25}\right)dx$$

$$= 4\ln|x^2+25| - 3\arctan\frac{x}{5} + C$$

3. Evaluate $\displaystyle\int \frac{4x+7}{2x-1}\,dx$.

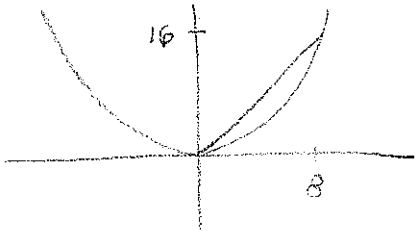
long division

$$\begin{array}{r|l} & 2 \\ \hline 2x-1 & 4x+7 \\ & \underline{4x-2} \\ & \quad 9 \end{array}$$

$$\int \frac{4x+7}{2x-1}\,dx = \int\left(2 + \frac{9}{2x-1}\right)dx$$

$$= 2x + \frac{9}{2}\ln|2x-1| + C$$

Let R be the region that lies between the curves $y = 2x$ and $y = x^2/4$. For problems 4–8, write down an expression involving an integral that represents the requested quantity. Do **NOT** evaluate the integrals.



$2x = \frac{x^2}{4}$

$x^2 - 8x = 0$

$y = 2x \qquad x = y/2$

$y = x^2/4 \qquad x = \sqrt{4y} = 2\sqrt{y}$

4. the volume generated when R is revolved around the x-axis

$$V = \int_0^8 \left(\pi (2x)^2 - \pi\left(\frac{x^2}{4}\right)^2\right) dx$$

5. the volume generated when R is revolved around the y-axis

$$V = \int_0^8 2\pi x\left(2x - \frac{x^2}{4}\right) dx$$

6. the volume generated when R is revolved around the line $x = 10$

$$V = \int_0^8 2\pi (10 - x)\left(2x - x^2/4\right) dx$$

7. the volume of the solid whose base is R and each cross-section of the solid taken perpendicular to the y-axis is a square

$$V = \int_0^{16} \left(2\sqrt{y} - y/2\right)^2 dy$$

8. the perimeter of R

$$P = \sqrt{8^2 + 16^2} + \int_0^8 \sqrt{1 + x^2/4}\, dx$$

9. Find the length of the curve $y = \frac{1}{12}x^3 + \frac{1}{x}$ on the interval $[1, 2]$.

$$y = \frac{1}{12}x^3 + \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{4}x^2 - \frac{1}{x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4} = \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2$$

$$L = \int_1^2 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)dx = \left(\frac{1}{12}x^3 - \frac{1}{x}\right)\Big|_1^2$$

$$= \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$$

The length of the curve is $\frac{13}{12}$.

$$x_1 + x_2 + \frac{40}{135}(h_1 + h_2)$$

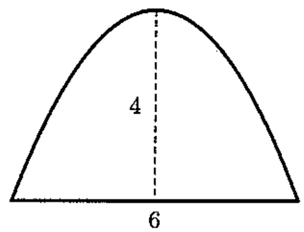
60	53, 53
57	51
50	41
29	37
17	

142 - 160
122 - 141
103 - 121
85 - 102
0 - 84

	48
53	43
45	38
37	33
30	

151
139
159
79
97

10. Set up an integral (you do not need to evaluate it) that gives the force exerted by a liquid on one side of the vertically submerged parabolic plate shown below. The units on the figure are feet and the top of the plate is eight feet beneath the surface of the liquid. Assume that the weight density of the liquid is 60 lb/ft^3.



$y = 4 - \frac{4}{9}x^2$ origin at base center

$x = \frac{3}{2}\sqrt{4-y}$ for right half

The force exerted by the liquid would be

$$F = \int_0^4 w(12-y) \cdot 3\sqrt{4-y}\, dy,$$

where $w = 60$. Thus

$$F = 180\int_0^4 (12-y)\sqrt{4-y}\, dy.$$

11. Evaluate $\int \frac{x-1}{x^2+x-6}\,dx$. partial fractions

$$\frac{x-1}{x^2+x-6} = \frac{x-1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$x-1 = A(x-2) + B(x+3)$$

$$x=-3 \Rightarrow -4 = -5A \Rightarrow A = 4/5$$

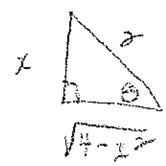
$$x=2 \Rightarrow 1 = 5B \Rightarrow B = 1/5$$

$$\int \frac{x-1}{x^2+x-6}\,dx = \int \left(\frac{4/5}{x+3} + \frac{1/5}{x-2} \right) dx$$

$$= \frac{4}{5} \ln|x+3| + \frac{1}{5} \ln|x-2| + C$$

12. Evaluate $\displaystyle\int_0^1 \frac{6x^2}{(4-x^2)^{5/2}}\,dx$.

use trig substitution



let $x = 2\sin\theta$

then $dx = 2\cos\theta\, d\theta$

$4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$

$$\int \frac{6x^2}{(4-x^2)^{5/2}}\,dx = \int \frac{24\sin^2\theta}{32\cos^5\theta}\cdot 2\cos\theta\, d\theta$$

$$= \frac{3}{2}\int \frac{\sin^2\theta}{\cos^4\theta}\, d\theta$$

$$= \frac{3}{2}\int \tan^2\theta \sec^2\theta\, d\theta$$

$$= \frac{3}{2}\cdot\frac{1}{3}\tan^3\theta + C$$

$$= \frac{1}{2}\left(\frac{x}{\sqrt{4-x^2}}\right)^3 + C$$

$$\int_0^1 \frac{6x^2}{(4-x^2)^{5/2}}\,dx = \frac{1}{2}\cdot\frac{x^3}{(4-x^2)^{3/2}}\Bigg|_0^1 = \frac{1}{2}\cdot\frac{1}{3\cdot\sqrt{3}} = \frac{\sqrt{3}}{18}.$$

or $$\int_0^1 \frac{6x^2}{(4-x^2)^{5/2}}\,dx = \frac{1}{2}\tan^3\theta\Bigg|_0^{\pi/6} = \frac{1}{2}\cdot\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{1}{6\sqrt{3}}.$$

Total: Name:

Math 126 **Third Exam** **Spring 2011**

Write neat, concise, and accurate solutions to each of the problems below—I will not give partial credit for steps I cannot follow. Include all relevant details and use correct notation. Calculators are not allowed for this exam. Each problem is worth six points.

1. Give an example (use formulas, not patterns) with the indicated properties. Be certain to use correct notation. No explanation is required but your example should be elementary.

   a) a convergent sequence that is not monotone

   $$\left\{ \frac{(-1)^n}{n} \right\}$$

   b) a monotone sequence that does not converge

   $$\left\{ n^2 \right\}$$

   c) a conditionally convergent series

   $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

2. Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^{k-1} - 2^k}{5^k}$.

$$\sum_{k=1}^{\infty} \frac{3^{k-1} - 2^k}{5^k} = \sum_{k=1}^{\infty} \frac{3^{k-1}}{5^k} - \sum_{k=1}^{\infty} \frac{2^k}{5^k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{3}{5}\right)^k - \sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^k$$

$$= \frac{1/5}{1 - 3/5} - \frac{2/5}{1 - 2/5}$$

$$= \frac{1}{2} - \frac{2}{3}$$

$$= -\frac{1}{6}$$

For problems 3–6, determine whether or not the given series converges. You must provide clear justification along with complete details for your conclusion. Be certain to mention which test you are using and pay careful attention to your notation.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{k^3+4k+5}$

For large values of k, the series resembles $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2}$, a convergent p-series. Since

$$\lim_{k\to\infty} \frac{2k+1}{k^3+4k+5} \div \frac{2}{k^2} = \lim_{k\to\infty} \frac{2k^3+k^2}{2k^3+8k+10} = 1,$$

the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{k^3+4k+5}$ converges by the Limit Comparison Test.

4. $\displaystyle\sum_{k=1}^{\infty} \frac{4^k\, k!}{2\cdot 7\cdot 12\cdot \cdots \cdot (5k+2)}$

We will use the Ratio Test.

$$\ell = \lim_{k\to\infty} \frac{4^{k+1}(k+1)!}{2\cdot 7\cdot 12\cdots(5k+2)(5k+7)} \cdot \frac{2\cdot 7\cdot 12\cdots(5k+2)}{4^k\, k!}$$

$$= \lim_{k\to\infty} \frac{4(k+1)}{5k+7}$$

$$= \frac{4}{5}$$

Since $\ell < 1$, the series converges.

5. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{7k+4}$

Since the sequence $\left\{\frac{1}{7k+4}\right\}$ is decreasing and converges to 0, the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{7k+4}$ converges by the Alternating Series Test.

6. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{2k^2+3k-1}}$

Since $\displaystyle\lim_{k\to\infty} \frac{k}{\sqrt{2k^2+3k-1}} = \lim_{k\to\infty} \frac{1}{\sqrt{2+3/k-1/k^2}} = \frac{1}{\sqrt{2}} \neq 0$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{2k^2+3k-1}}$ diverges by the Divergence Test.

7. Find the limit of the sequence $\left\{\left(\frac{3n}{3n+1}\right)^n\right\}$. You must show how you arrived at your answer.

$$\left(\frac{3n}{3n+1}\right)^n = \left(\frac{3n+1}{3n}\right)^{-n} = \left(\left(1+\frac{1/3}{n}\right)^n\right)^{-1}$$

By a theorem $\left\{\left(1+\frac{1/3}{n}\right)^n\right\}$ converges to $e^{1/3}$ so

$\left\{\left(\frac{3n}{3n+1}\right)^n\right\}$ converges to $e^{-1/3}$.

8. For each positive integer n, let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{10n}$. Prove that the sequence $\{x_n\}$ is bounded.

For each positive integer n,

$$\begin{aligned} 0 < x_n &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+9n} \\ &< \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\ &= \frac{9n}{n} \\ &= 9. \end{aligned}$$

Since all of the terms are between 0 and 9, the sequence $\{x_n\}$ is bounded.

9. Find the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{k^2}{3^k}(x+2)^k$

Using the Root Test, we find that

$$l = \lim_{k\to\infty} \sqrt[k]{\left|\frac{k^2}{3^k}(x+2)^k\right|} = \lim_{k\to\infty} \left(\sqrt[k]{k}\right)^2 \cdot \frac{|x+2|}{3} = \frac{|x+2|}{3}$$

The power series converges when $|x+2| < 3$ so the radius of convergence is 3.

10. Let $\{f_n\}$ denote the Fibonacci sequence. Use the Principle of Mathematical Induction to prove that $\sum_{i=1}^{2n+1}(-1)^{i-1}f_i = f_{2n}+1$ for each positive integer n.

We will use the PMI. For $n=1$, we have

$\sum_{i=1}^{3}(-1)^{i-1}f_i = f_1 - f_2 + f_3 = 1-1+2 = 2$ and $f_2+1=2$,

so the equation is valid when $n=1$. Suppose that

$\sum_{i=1}^{2k+1}(-1)^{i-1}f_i = f_{2k}+1$ for some positive integer k. Then

$$\sum_{i=1}^{2(k+1)+1}(-1)^{i}f_i = \sum_{i=1}^{2k+1}(-1)^{i-1}f_i + (-1)^{2k+1}f_{2k+2} + (-1)^{2k+2}f_{2k+3}$$

$$= f_{2k}+1 - f_{2k+2} + f_{2k+3}$$

$$= f_{2k}+1+f_{2k+1}$$

$$= f_{2k+2}+1 = f_{2(k+1)}+1,$$

which is the desired equation for $n=k+1$. By the PMI,

$\sum_{i=1}^{2n+1}(-1)^{i-1}f_i = f_{2n}+1$ for all positive integers n.

212 - 240
182 - 211
152 - 181

233 $\mathcal{S}$
221 $\mathcal{T}$
200 m

52 57
44 51
36 45
28 39

54
48
42
36