# Chapter 1

## The Foundation of Euclidean Geometry

"This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state." — Clifford

### 1. Introduction.

Geometry, that branch of mathematics in which are treated the properties of figures in space, is of ancient origin. Much of its development has been the result of efforts made throughout many centuries to construct a body of logical doctrine for correlating the geometrical data obtained from observation and measurement. By the time of Euclid (about 300 B.C.) the science of geometry had reached a well-advanced stage. From the accumulated material Euclid compiled his *Elements*, the most remarkable textbook ever written, one which, despite a number of grave imperfections, has served as a model for scientific treatises for over two thousand years.

Euclid and his predecessors recognized what every student of philosophy knows: that not everything can be proved. In building a logical structure, one or more of the propositions must be assumed, the others following by logical deduction. Any attempt to prove all of the propositions must lead inevitably to the completion of a vicious circle. In geometry these assumptions originally took the form of postulates suggested by experience and intuition. At best these were statements of what seemed from observation to be true or approximately true. A geometry carefully built upon such a foundation may be expected to correlate the data of observation very well, perhaps, but certainly not exactly. Indeed, it should be clear that the mere change of some more-or-less doubtful postulate of one geometry may lead to another geometry which, although radically different from the first, relates the same data quite as well. We shall, in what follows, wish principally to regard geometry as an abstract science, the postulates as mere assumptions. But the practical aspects are not to be ignored. They have played no small role in the evolution of abstract geometry and a consideration of them will frequently throw light on the significance of our results and help us to determine whether these results are important or trivial.

In the next few paragraphs we shall examine briefly the foundation of Euclidean Geometry. These investigations will serve the double purpose of introducing the Non-Euclidean Geometries and of furnishing the background for a good understanding of their nature and significance.

### 2. The Definitions.

The figures of geometry are constructed from various elements such as points, lines, planes, curves, and surfaces. Some of these elements, as well as their relations to each other, must be left undefined, for it is futile to attempt to define all of the elements of geometry, just as it is to prove all of the propositions. The other elements and relations are then defined in terms of these fundamental ones. In laying the foundation for his geometry, Euclid[1] gave twenty-three definitions.[2] A number of these might very well have been omitted. For example, he defined a point as *that which has no part*; a line, according to him, is *breadthless length*, while a plane surface is one which *lies evenly with the straight lines on itself.* From the logical viewpoint, such definitions as these are useless. As a matter of fact, Euclid made no use of them. In modern geometries, point, line, and plane are not defined directly; they are described by being restricted to satisfy certain relations, defined or undefined, and certain postulates. One of the best of the systems constructed to serve as a logical basis for Euclidean Geometry is that of Hilbert.[3] He begins by considering three classes of *things*, points, lines, and planes. "We think of these points, straight lines, and planes," he explains, "as having certain mutual relations, which we indicate by such words as *are situated between, parallel, congruent, continuous*, etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry."

The majority of Euclid's definitions are satisfactory enough. Particular attention should be given to the twenty-third, for it will play an important part in what is to follow. It is the definition of parallel lines the best one, viewed from an elementary standpoint, ever devised.

*Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

In contrast with this definition, which is based on the concept of parallel lines *not meeting*, it seems important to call attention to two other concepts which have been used extensively since ancient times.[4] These involve the ideas that two parallel lines are lines which have the *same direction* or which are everywhere *equally distant*. Neither is satisfactory.

The *direction*-theory leads to the completion of a vicious circle. If the idea of *direction* is left undefined, there can be no test to apply to determine whether two given lines are parallel. On the other hand, any attempt to define *direction* must depend upon some knowledge of the behavior of parallels and their properties.

The *equidistant*-theory is equally unsatisfactory. It depends upon the assumption that, for the particular geometry under consideration, the locus of points equidistant from a straight line is a straight line. But this must be proved, or at least shown to be compatible with the other assumptions. Strange as it may seem, we shall shortly encounter geometries in which this is not true.

Finally, it is worth emphasizing that, according to Euclid, two lines in a plane *either meet or are parallel.* There is no other possible relation.

### 3. The Common Notions.

The ten assumptions of Euclid are divided into two sets: five are classified as *common notions*, the others as *postulates*. The distinction between them is not thoroughly clear. We do not care to go further than to remark that the common notions seem to have been regarded as assumptions acceptable to all sciences or to all intelligent people, while the postulates were considered as assumptions peculiar to the science of geometry. The five common notions are:

1. *Things which are equal to the same thing are also equal to one another.*
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

One recognizes in these assumptions propositions of the type which at one time were so frequently described as "self-evident." From what has already been said, it should be clear that this is not the character of the assumptions of geometry at all. As a matter of fact, no "self-evident" proposition has ever been found.

### 4. The Postulates.

Euclid postulated the following:

1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a circle with any center and distance.*
4. *That all right angles are equal to one another.*
5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Although Euclid does not specifically say so, it seems clear that the First Postulate carries with it the idea that the line joining two points is *unique* and that two lines cannot therefore enclose a space. For example, Euclid tacitly assumed this in his proof of I.4.[5] Likewise it must be inferred from the Second Postulate that the finite straight line can be produced at each extremity in only one way, so that two different straight lines cannot have a common segment. Explicit evidence of this implication first appears in the proof of XI.1, although critical examination shows that it is needed from the very beginning of Book I. In regard to the Third Postulate, we merely remark that the word *distance* is used in place of radius, implying that each point of the circumference is at this distance from the center. The Fourth Postulate provides a standard or unit angle in terms of which other angles can be measured. Immediate use of this unit is made in Postulate 5.

The Fifth[6] Postulate plays a major role in what follows. In fact it is the starting point in the study of Non-Euclidean Geometry. One can hardly overestimate the effect which this postulate, together with the controversies which surrounded it, has had upon geometry, mathematics in general, and logic. It has been described [7] as "perhaps the most famous single utterance in the history of science." On account of its importance, we shall return to it soon and treat it at length.

### 5. Tacit Assumptions Made by Euclid. Superposition.

In this and the remaining sections of the chapter we wish to call attention to certain other assumptions made by Euclid. With the exception of the one concerned with superposition, they were probably made unconsciously; at any rate they were not stated and included among the common notions and postulates. These omissions constitute what is regarded by geometers as one of the gravest defects of Euclid's geometry.

Euclid uses essentially the same proof for Proposition I.4 that is used in most modern elementary texts. There is little doubt that, in proving the congruence of two triangles having two sides and the included angle of one equal to two sides and the included angle of the other, he actually regarded one triangle as being moved in order to make it coincide with the other. But there are objections to such recourse to the idea of motion without deformation in the proofs of properties of figures in space.[8] It appears that Euclid himself had no high regard for the method and used it reluctantly.

Objections arise, for example, from the standpoint that points are *positions* and are thus incapable of motion. On the other hand, if one regards geometry from the viewpoint of its application to physical space and chooses to consider the figures as capable of displacement, he must recognize that the material bodies which are encountered are always more-or-less subject to distortion and change. Nor, in this connection, may there be ignored the modern physical concept that the dimensions of bodies in motion are not the same as when they are at rest. However, in practice, it is of course possible to make an approximate comparison of certain material bodies by methods which resemble superposition. This may suggest the formulation in geometry of a postulate rendering superposition legitimate. But Euclid did not do this, although there is evidence that he may have intended Common Notion 4 to authorize the method. In answer to the objections, it also may be pointed out that what has been regarded as motion in superposition is, strictly speaking, merely a transference of attention from one figure to another.

The use of superposition can be avoided. Some modern geometers do this, for example, by *assuming* that, if two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, the remaining pairs of corresponding angles are equal.[9]

**6. The Infinitude of the Line.**

Postulate 2, which asserts that a straight line can be produced continuously, does not necessarily imply that straight lines are infinite. However, as we shall discover directly, Euclid unconsciously assumed the infinitude of the line.

It was Riemann who first suggested the substitution of the more general postulate that the straight line is *unbounded*. In his remarkable dissertation, *Über die Hypothesen welche der Geometrie zu Grundeliegen*,[10] read in 1854 to the Philosophical Faculty at Göttingen, he pointed out that, however certain we may be of the unboundedness of space, we need not as a consequence infer its infinitude. He said,[11] "In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*; the former belongs to the extent relations, the latter to the measure relations. That space is an unbounded threefold manifoldness is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is forever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value."

We shall learn later that geometries, logically as sound as Euclid's, can be constructed upon the hypothesis that straight lines are boundless, being closed, but not infinite. In attempting to conceive straight lines of this character, the reader may find it helpful, provided he does not carry the analogy too far, to consider the great circles of a sphere. It is well known that in spherical geometry the great circles are *geodesics*, i.e., they are the "lines" of shortest distance between points. It will not be difficult to discover that they have many other properties analogous to those of straight lines in Euclidean Plane Geometry. On the other hand there are many striking differences. We note, for example, that these "lines," while endless, are not infinite; that, while in general two points determine a "line," two points may be so situated that an infinite number of "lines" can be drawn through them; that two "lines" always intersect in two points and enclose a space.

Even a cursory consideration of the consequences of attributing to straight lines the character of being boundless, but not infinite, convinces one that Euclid tacitly assumed the infinitude of the line. One critical point at which this was done is found in the proof of I.16. This proposition is of such importance in what follows and its consequences are so far-reaching that we present the proof here.

Proposition I.16: In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let $ABC$ (Fig. 1) be the given triangle, with $BC$ produced to $D$. We shall prove that $\angle ACD > \angle BAC$. Let $E$ be the midpoint of $AC$. Draw $BE$ and produce it to $F$, making $EF$ equal to $BE$. Draw $CF$. Then triangles $BEA$ and $FEC$ are congruent and consequently angles $FCE$ and $BAC$ are equal. But $\angle ACD > \angle FCE$. Therefore $\angle ACD > \angle BAC$.

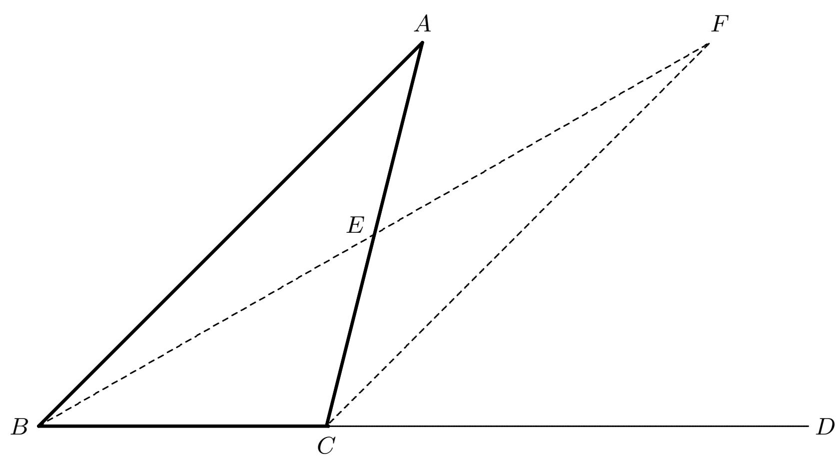


Figure 1

It should be clear now that this proof may fail, if the straight line is not infinite. As a matter of fact, the same proof can be used in spherical geometry for a spherical triangle but is valid only so long as $BF$ is less than a semicircle. If $F$ lies on $CD$, angle $ACD$ is equal to angle $ECF$ and consequently to angle $BAC$. If $BF$ is greater than a semicircle, angle $ACD$ will be less than angle $BAC$. Even if one conceives a geometry in which any two of the closed lines intersect in only one point, $BF$ may be so long that $F$ will coincide with $B$ or lie on segment $BE$. In either case the proof fails.

The proofs of a number of important propositions in Euclidean Geometry depend upon I.16. Such propositions as I.17, 18, 19, 20, and 21 will not be valid without restrictions when I.16 does not hold.

## Exercise

Prove Propositions I.17, 18, 19, 20, 21.

## 7. Pasch's Axiom.

Another important assumption made by Euclid, without explicit statement, has been formulated by Pasch[12] as follows.

*Let $A$, $B$, $C$ be three points not lying in the same straight line and let $\alpha$ be a straight line lying in the plane of $ABC$ and not passing through any of points $A$, $B$, or $C$. Then, if the line $\alpha$ passes through a point of the segment $AB$, it will also pass through a point of the segment $BC$ or a point of the segment $AC$.*

It readily follows from this that, if a line enters a triangle at a vertex, it must cut the opposite side. Euclid tacitly assumed this frequently as, for example, in the proof of I.21.

We shall make use of Pasch's Axiom many times in what follows at points where intuition cannot be depended upon to guide us as safely as it did in Euclid. In order to emphasize the importance of an explicit statement of this axiom as a characteristic of Euclidean Geometry, we remark that there are geometries in which it holds only with restrictions. It will be recognized that it is true for spherical triangles only if they are limited in size.

Pasch's Axiom is one of those assumptions classified by modern geometers as *axioms of order.*[13] These important axioms bring out the idea expressed by the word *between* and make possible an *order of sequence* of the points on a straight line.

## 8. The Principle of Continuity.

One of the features of Euclid's geometry is the frequent use of constructions to prove the existence of figures having designated properties. The very first proposition is of this type and the reader will have no difficulty in recalling others. In these constructions, lines and circles are drawn, and the points of intersection of line with line, line with circle, circle with circle, are assumed to exist. Obviously, in a carefully constructed geometry, the existence of these points must be postulated or proved.

The only one of Euclid's postulates which does anything like this is the Fifth, and it applies only to a particular situation. What is needed is a postulate which will ascribe to all lines and circles that characteristic called *continuity.* This is done in a satisfactory way by one due to Dedekind.[14]

The Postulate of Dedekind. *If all points of a straight line fall into two classes, such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.*

"I think I shall not err," remarks Dedekind, "in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is revealed. To this I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in a line. If space has at all a real existence, it is *not* necessary for it to be continuous; many of its properties would be the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle."

This postulate can easily be extended to cover angles and arcs as well as linear segments. As an application of the postulate we shall prove the following proposition:

*The segment of line joining a point inside a circle to a point outside the circle has a point in common with the circle.*

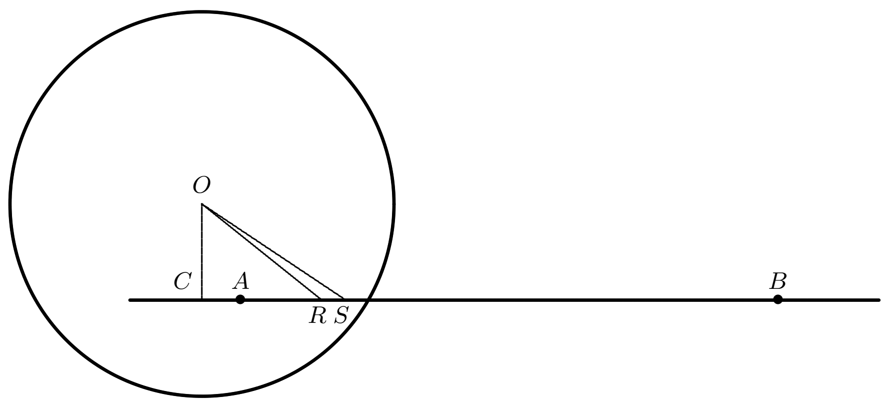


Figure 2

Let $O$ be the center of the given circle (Fig. 2) and $r$ its radius; let $A$ be the point inside and $B$ the point outside. Then $OA < r < OB$. Draw $OC$ perpendicular to $AB$, produced if necessary, and note that $OC \leq OA < r$. The points of segment $AB$ can now be divided into two classes: those points $P$ for which $OP < r$ and those points $Q$ for which $OQ \geq r$. Since, in every case, $OP < OQ$, it follows that $CP < CQ$, and thus every point $P$ precedes (or follows) every point $Q$. Hence, by the Postulate of Dedekind, there exists a point $R$ of the segment $AB$ such that all points which precede it belong to one class and all which follow it belong to the other class. We proceed to prove by *reductio ad absurdum* that $R$ is on the circle.

Assume that $OR < r$ and choose point $S$ on $AB$, between $R$ and $B$, such that $RS < r - OR$. Since, in triangle $ORS$, $OS < OR + RS$, we conclude that $OS < r$. But this is absurd, and consequently $OR$ cannot be less than $r$.

The reader can easily show in a similar way that $OR$ cannot be greater than $r$.

The idea of continuity is frequently introduced into geometry through what is known as the *Postulate of Archimedes*. A simple, but quite satisfactory, statement of the postulate is as follows:

*Given two linear segments, there is always some finite multiple of one which is greater than the other.*

This can be shown[15] to be a consequence of the Postulate of Dedekind. One observes that it prescribes the exclusion of both infinite and infinitesimal segments. It holds for arcs and angles as well as line segments. We shall make use of it upon several occasions.

A large portion of Euclidean Geometry and also of Non-Euclidean Geometry can be constructed without the employment of the Principle of Continuity. We shall, however, make no particular effort to avoid its use in what follows.

**9. The Postulate System of Hilbert.**

The work of such men as Pasch, Veronese, Peano, and Hilbert has placed Euclidean Geometry on a sound, logical basis. It will be helpful to conclude this chapter by giving the system of postulates, slightly abbreviated in form, set down by Hilbert—the system referred to in Section 2. They are arranged in six sets. It will be recalled that Hilbert begins with the undefined elements *point*, *line*, and *plane*. These elements are characterized by certain relations which are described in the postulates.

I. *The Postulates of Connection.*

1 and 2. *Two distinct points determine one and only one straight line.*

3. *There are at least two points on every line, and there are at least three points on every plane which do not lie on the same straight line.*

4 and 5. *Three points which do not lie on the same straight line determine one and only one plane.*

6. *If two points of a line lie on a plane, then all points of the line lie on the plane.*

7. *If two planes have one point in common, they have at least one other point in common.*

8. *There exist at least four points which do not lie on the same plane.*

Among the theorems to be deduced from the above set of postulates are the following:

Two distinct straight lines lying on a plane have one point or no point in common.

A line and a point not lying on that line determine a plane; so also do two distinct lines which have a point in common.

II. *The Postulates of Order.*

The postulates of this set describe an undefined relation among the points of a straight line a relation expressed by the word *between*.

1. *If $A$, $B$, and $C$ are points of a straight line and $B$ is between $A$ and $C$, then $B$ is also between $C$ and $A$.*
2. *If $A$ and $C$ are two points of a straight line, there exists at least one other point of the line which lies between them.*
3. *Of any three points of a straight line, one and only one lies between the other two.*

Two points $A$ and $B$ determine a *segment*; $A$ and $B$ are the *ends* of the segment and the points between $A$ and $B$ are the *points of the segment*.

4. *Given three points $A$, $B$, and $C$, which are not on the same straight line, and a straight line in the plane of $ABC$ not passing through any of the points $A$, $B$, or $C$, then if the line contains a point of the segment $AB$, it also contains a point either of the segment $BC$ or of the segment $AC$.* (Pasch's Axiom.)

As deductions from the postulates already stated, we note the following:

Between any two points of a straight line there is always an unlimited number of points.

Given a finite number of points on a straight line, they can always be considered in a sequence $A, B, C, D, E, \ldots, K$, such that $B$ lies between $A$ and $C, D, E, \ldots, K$; $C$ lies between $A$, $B$ and $D, E, \ldots, K$; etc. There is only one other sequence with the same properties, namely, the reverse, $K, \ldots, E, D, C, B, A$.

Every straight line of a plane divides the points of the plane which are not on the line into two regions with the following properties: Every point of one region determines with every point of the other a segment containing a point of the line; on the other hand, any two points of the same region determine a segment not containing a point of the line. Thus we say that two points are on *the same side* of a line or on *opposite sides*. In a similar way, a given point of a line divides the points of a line into *half-lines* or *rays*, each ray consisting of all points of the line on *one side* of the given point.

A system of segments $AB$, $BC$, $CD$,…,$KL$ is called a *broken line* joining $A$ to $L$. The points $A$, $B$, $C$, $D$,…,$L$, as well as the points of the segments, are called the points of the broken line. If $A$ and $L$ coincide, the broken line is called a *polygon*. The segments are called the *sides* of the polygon, and the points $A$, $B$, $C$, $D$, …, $K$ are called the *vertices*. Polygons having $3, 4, 5, \dots, n$ vertices are called, respectively, triangles, quadrangles, pentagons, …, $n$-gons. If the vertices of a polygon are distinct and none lies on a side, and if no sides have a point in common, the polygon is called a *simple polygon*.

It follows that every simple polygon which lies in a plane divides the points of the plane not belonging to the polygon into two regions—an *interior* and an *exterior*—having the following properties: A point of one region cannot be joined to a point of the other by a broken line which does not contain a point of the polygon. Two points of the same region can, however, be so joined. The two regions can be distinguished from one another by the fact that there exist lines in the plane which lie entirely outside the polygon, but there is none which lies entirely within it.

III. *The Postulates of Congruence.*

This set of postulates introduces a new concept designated by the word congruent.

1. *If $A$ and $B$ are two points of a straight line $\ell$ and $A'$ is a point on the same or another straight line $\ell'$, then there exists on $\ell'$, on a given side of $A'$, one and only one point $B'$ such that the segment $AB$ is congruent to the segment $A'B'$. Every segment is congruent to itself.*
2. *If a segment $AB$ is congruent to a segment $A'B'$ and also to another segment $A''B''$, then $A'B'$ is congruent to segment $A''B''$.*
3. *If segments $AB$ and $BC$ of a straight line $\ell$ have only point $B$ in common, and if segments $A'B'$ and $B'C'$ of the same or another straight line $\ell'$ have only $B'$ in common, then if $AB$ and $BC$ are, respectively, congruent to $A'B'$ and $B'C'$, $AC$ is congruent to $A'C'$.*

The system of two rays $h$ and $k$ emanating from a point $O$ and lying on different lines is called an *angle* $(h, k)$. The rays are called *the sides* of the angle and the point $O$ its *vertex*. It can be proved that an angle divides the points of its plane, excluding $O$ and the points on the sides, into two regions. Any two points of either region can always be joined by a broken line containing neither $O$ nor any point of either side, while no point of one region can be so joined to a point of the other. One region, called the *interior* of the angle, has the property that the segment determined by any two of its points contains only points of the region; for the other region, called the *exterior*, this is not true for every pair of points.

4. *Given an angle $(h, k)$ on plane $\alpha$, a line $\ell'$ on the same or a different plane $\alpha'$, a point $O'$ on $\ell'$, and on line $\ell'$ a ray $h'$ emanating from $O'$, then on $\ell'$ and emanating from $O'$ there is one and only one ray $k'$ such that the angle $(h', k')$ is congruent to angle $(h, k)$ and the interior of $(h', k')$ is on a given side of $\ell'$.*
5. *If the angle $(h, k)$ is congruent to angle $(h', k')$ and also to angle $(h'', k'')$, then angle $(h', k')$ is congruent to angle $(h'', k'')$.*

The last two postulates characterize angles in the same way that III. 1 and 2 characterize segments. The final postulate of this set relates the congruence of segments and the congruence of angles.

6. *If, for triangles* $ABC$ *and* $A'B'C'$, $AB$, $AC$ *and angle* $BAC$ *are, respectively, congruent to* $A'B'$, $A'C'$ *and angle* $B'A'C'$, *then the angle* $ABC$ *is congruent to angle* $A'B'C'$.

IV. The Postulate of Parallels.

*Given a line* $\ell$ *and a point* $A$ *not lying on* $\ell$, *then there exists, in the plane determined by* $\ell$ *and* $A$, *one and only one line which contains* $A$ *but not any point of* $\ell$. (Playfair's Axiom.)

V. The Postulate of Continuity.

*Given any two segments* $AB$ *and* $CD$, *there always exists on the line* $AB$ *a sequence of points* $A_1$, $A_2$, $A_3$, $\ldots$, $A_n$, *such that the segments* $AA_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ *are congruent to* $CD$ *and* $B$ *lies between* $A$ *and* $A_n$. (Postulate of Archimedes.)

VI. The Postulate of Linear Completeness.

*It is not possible to add, to the system of points of a line, points such that the extended system shall form a new geometry for which all of the foregoing linear postulates are valid.*

Upon this foundation rests the geometry which we know as *Euclidean*.

### Footnotes

1. In this book, all specific statements pertaining to Euclid's text and all quotations from Euclid are based upon T. L. Heath's excellent edition: *The Thirteen Books of Euclid's Elements*, 2nd edition (Cambridge, 1926). By permission of The Macmillan Company.
2. These definitions are to be found in the Appendix.
3. *Grundlagen der Geometrie*, 7th edition (Leipzig and Berlin, 1930), or *The Foundations of Geometry*, authorized translation of the 1st edition by E. J. Townsend (Chicago, 1902). All references will be to the former unless the translation or another edition is specified. See Section 9 for this postulate system.
4. Heath, *loc. cit.*, Vol. I, p. 190, ff.
5. The propositions of Book I are to be found, stated without proof, in the Appendix.
6. This postulate is also sometimes referred to as the Eleventh or Twelfth.
7. Keyser, *Mathematical Philosophy* (New York, 1922).
8. See Heath, *loc. cit.*, Vol. I, pp. 224–228.
9. See, for example, Hilbert, *loc. cit.*, and also Section 9.
10. Riemann: *Gesammelte Mathematische Werke* (Leipzig, 1892).
11. This quotation is from a translation by W. K Clifford, in *Nature*, Vol. VIII, 1873. A translation by H. S. White is to be found in David Eugene Smith's *A Source Book in Mathematics* (New York, 1929).
12. Pasch, *Vorlesungen über neuere Geometrie* (Berlin, 1926).
13 See Section 9.
14. Dedekind, *Essays on the Theory of Numbers*, authorized translation by W. W. Beman (Chicago, 1901), or *Gesammelte Mathematische Werke*, Vol. III, p. 322 (Brunswick, 1932).
15. See the paper by G. Vitali in Enriques' collection, *Questioni riguardanti la geometria elementare* (Bologna, 1900), or a translation into German, *Fragen der Elementargeometrie*, Vol. I, p. 135 (Leipzig and Berlin, 1911).