Chapter 3

The Discovery of Non-Euclidean Geometry

"Out of nothing I have created a strange new universe." — Johann Bolyai

28. Introduction.

The beginning of the nineteenth century found the obstinate puzzle of the Fifth Postulate still unsolved. But one should not get the impression that the efforts to prove the Postulate, made throughout more than twenty centuries, were altogether fruitless. Slowly but surely they had directed the speculations of geometers to the point where the discovery of Non-Euclidean Geometry could not long be delayed. In retrospect, one wonders at first that this preparation should have taken so long, but on second thought marvels that such a momentous discovery came as early as it did. At the time the new ideas were crystalizing, the philosophy of Kant (1714–1804) dominated the situation, and this philosophy treated space not as empirical, but as intuitive. From this viewpoint, space was regarded as something already existing in the mind and not as a concept resulting from external experience. In that day it required not only perspicacity, but courage, to recognize that geometry becomes an experimental science, once it is applied to physical space, and that its postulates and their consequences need only be accepted if convenient and if they agree reasonably well with experimental data.

But the change of viewpoint gradually came. The discovery of Non-Euclidean Geometry led eventually to the complete destruction of the Kantian space conception and at last revealed not only the true distinction between concept and experience but, what is even more important, their interrelation.

We are not surprised that, when the time came, the discovery of Non-Euclidean Geometry was not made by one man, but independently by several in different parts of the world. This has happened more than once in the history of mathematics and it will doubtless happen again. The father of Johann Bolyai, one of the founders of Non-Euclidean Geometry, predicted this when, in a letter to his son urging that he make public his discoveries without delay, he wrote,[1] "It seems to me advisable, if you have actually succeeded in obtaining a solution of the problem, that, for a twofold reason, its publication be hastened: first, because ideas easily pass from one to another who, in that case, can publish them; secondly, because it seems to be true that many things have, as it were, an epoch in which they are discovered in several places simultaneously, just as the violets appear on all sides in springtime." And so it happened that independently and at about the same time the discovery of a logically consistent geometry, in which the Fifth Postulate was denied, was made by Gauss in Germany, Bolyai in Hungary and Lobachewsky in Russia.

29. Gauss.

At the turn of the century, during those critical years in the evolution of geometry, the dominant figure in the mathematical world was Carl Friedrich Gauss (1777–1855). Naturally he took no small part in the development of the ideas which led to the discovery of the new systems of geometry. Few of the results of his many years of meditation and research on the problems associated with the Fifth Postulate were published or made public during his lifetime. Some letters written to others interested in those problems, two published reviews of certain tracts on parallels and a few notes discovered among his papers furnish meager but sufficient evidence that he was probably the first to understand clearly the possibility of a logically sound geometry different from Euclid's. It was he who first called the new geometry Non-Euclidean. The correspondence and reviews[2] referred to outline rather clearly the progress he made in the study of parallels, and show that recognition of the new geometry did not come suddenly but only after many years of thought.

It seems clear that, even as late as the first decade of the new century, Gauss, traveling in the footsteps of Saccheri and Lambert, with whose books he may have been familiar, was still attempting to prove the Fifth Postulate by the *reductio ad absurdum* method, but that he fully recognized the profound character of the obstacles encountered. It was during the second decade that he began the formulation of the idea of a new geometry, to develop the elementary theorems and to dispel his doubts. No words can describe the nature of his discoveries, the significance he attached to them, his attitude toward the current concept of space and his fear of being misunderstood, half so well as his own words in a letter written at Göttingen on November 8, 1814 to F. A. Taurinus. The following is a translation of this important document.[3] "I have not read without pleasure your kind letter of October 30th with the enclosed abstract, all the more because until now I have been accustomed to find little trace of real geometrical insight among the majority of people who essay anew to investigate the so-called Theory of Parallels.

"In regard to your attempt, I have nothing (or not much) to say except that it is incomplete. It is true that your demonstration of the proof that the sum of the three angles of a plane triangle cannot be greater than 180° is somewhat lacking in geometrical rigor. But this in itself can easily be remedied, and there is no doubt that the impossibility can be proved most rigorously. But the situation is quite different in the second part, that the sum of the angles cannot be less than 180°; this is the critical point, the reef on which all the wrecks occur. I imagine that this problem has not engaged you very long. I have pondered it for over thirty years, and I do not believe that anyone can have given more thought to this second part than I, though I have never published anything on it. The assumption that the sum of the three angles is less than 180° leads to a curious geometry, quite different from ours (the Euclidean), but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of the determination of a constant, which cannot be designated *a priori*. The greater one takes this constant, the nearer one comes to Euclidean Geometry, and when it is chosen infinitely large the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit, regardless of how great the sides are taken, nor indeed can it ever reach it. All my efforts to discover a contradiction, an inconsistency, in this Non-Euclidean Geometry have been without success, and the one thing in it which is opposed to our conceptions is that, if it were true, there must exist in space a linear magnitude, *determined for itself* (but unknown to us). But it seems to me that we know, despite the say-nothing word-wisdom of the metaphysicians, too little, or too nearly nothing at all, about the true nature of space, to consider as *absolutely impossible* that which appears to us unnatural. If this Non-Euclidean Geometry were true, and it were possible to compare that constant with such magnitudes as we encounter in our measurements on the earth and in the heavens, it could then be determined *a posteriori*. Consequently in jest I have sometimes expressed the wish that the Euclidean Geometry were not true, since then we would have *a priori* an absolute standard of measure.

"I do not fear that any man who has shown that he possesses a thoughtful mathematical mind will misunderstand what has been said above, but in any case consider it a private communication of which no public use or use leading in any way to publicity is to be made. Perhaps I shall myself, if I have at some future time more leisure than in my present circumstances, make public my investigations."

The failure of Gauss to make public his results made it inevitable that the world withhold a portion of the honor which might have been entirely his. As we shall see, others who came to the same conclusions, although probably a little later, promptly extended the ideas and courageously published them. To accord to these the glory in its fullest form is only just. But one cannot fail altogether to sympathize with Gauss in his reluctance to divulge his discoveries. By his day many prominent mathematicians, dominated by the philosophy of Kant, had come to the conclusion that the mystery of the Fifth Postulate could never be solved. There were still those who continued their investigations, but they were likely to be regarded as cranks. It was probably the derision of smug and shallow-minded geometers that Gauss feared. Nor can one safely say that he had less courage than those who made public their results. Compared to him, they were obscure, with no reputations to uphold and nothing much to lose. Gauss, on the other hand, had climbed high. If he fell, he had much farther to fall.

In a letter[4] to Schumacher, dated May 17, 1831, and referring to the problem of parallels, Gauss wrote : "I have begun to write down during the last few weeks some of my own meditations, a part of which I have never previously put in writing, so that already I have had to think it all through anew three or four times. But I wished this not to perish with me."

Consequently, among his papers there is to be found a brief account of the elementary theory of parallels for the new geometry. We have already noted that one of the simplest substitutes for the Fifth Postulate is the so-called Playfair Axiom. In rejecting the Postulate Gauss, like Bolyai and Lobachewsky, chose to assume that through a point more than one parallel (in the sense of Euclid) can be drawn to a given line.

There is no need to sketch through the details of what little he jotted down; it is essentially similar to the elementary theory presented in the first few pages of the next chapter. He did not go far in recording his meditations; his notes came to a sudden halt. For on February 14, 1831 he received a copy of the famous *Appendix* by Johann Bolyai.

30. Bolyai.

While studying at Göttingen, Gauss numbered among his friends a Hungarian, Wolfgang Bolyai[5] (Bolyai Farkas, 1775–1856), who was a student there from 1796 to 1799. It is quite certain that the two frequently discussed problems related to the theory of parallels. After they left the University, they continued their intercourse by correspondence. A letter[6] written by Gauss to Bolyai in 1799 shows that both were at that time still attempting to prove the Fifth Postulate. In 1804, Bolyai, convinced that he had succeeded in doing this, presented his ideas in a little tract entitled *Theoria Parallelarum*,[7] which he sent to Gauss, enclosed with a letter. But the proof was incorrect, and Gauss, in replying, pointed out the error. Undaunted, Bolyai continued to reason along the same lines and, four years later, sent to Gauss a supplementary paper.[8] He apparently became discouraged when Gauss did not reply, and turned his attention to other matters. However, during the next two decades, despite varied interests as professor, poet, dramatist, musician, inventor and forester, he managed to collect his ideas on elementary mathematics and finally publish them in 1831–33 in a two volume work which we shall call briefly the *Tentamen*.[9] Wolfgang Bolyai was a talented and capable man, but his claim to fame must doubtless be based upon the fact that he was the father of Johann. For on December 15, 1801 was born Johann Bolyai (Bolyai Janos, 1802–1860). "He is, Heaven be praised," wrote Wolfgang to Gauss in 1803, "a healthy and very beautiful child, with a good disposition, black hair and eyebrows, and burning deep blue eyes, which at times sparkle like two jewels." And during those years leading up to the publication of the *Tentamen*, Johann had been growing to manhood.

His father gave him his early instruction in mathematics, so that it does not seem unnatural that he should have become interested in the theory of parallels. Nor is it a matter for surprise to learn that, by the time he had become a student in the Royal College for Engineers at Vienna in 1817, he had devoted much thought to the problem of the proof of the Fifth Postulate, despite the fact that his father, recalling his own unsuccessful efforts, recommended that the ancient enigma was something to be left entirely alone. But, by 1810, his efforts to prove the Postulate by the substitution of a contradictory assumption began to yield results of a different nature. His attention was gradually directed toward the possibility of formulating a general geometry, an Absolute Science of Space, with Euclidean Geometry as a special case.

In his attempts to prove the Fifth Postulate by denying it, Bolyai chose to regard that assumption in the form which we have already designated as Playfair's Axiom, and which asserts that one and only one parallel line can be drawn through a given point to a given line. The denial of the Postulate then implies either that no parallel to the line can be drawn through the point or that more than one such parallel can be drawn. But, as a consequence of Euclid I, 27 and 28, provided the straight line is regarded as infinite, the former of the two implications must be discarded. Furthermore, if there are at least two parallels to the line through the point, then there must be an infinite number of parallels in the sense of Euclid.

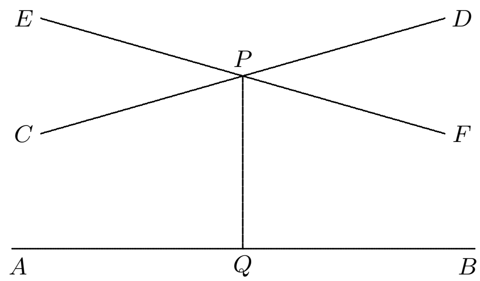

Figure 20

If, for example, the two lines CD and EF (Fig. 20) through P do not cut AB, then the same will be true for all lines through P which lie within the vertical angles EPC and DPF. In substance Bolyai, as did Gauss and Lobachewsky, then argued that if one starts with PQ perpendicular to AB and allows PQ to rotate about P in either direction, it will continue to cut AB awhile and then cease to cut it. He was thus led to postulate the existence of two lines through P which separate the lines which cut AB from those which do not. Since for rotation of PQ in either direction there is no last cutting line, these postulated lines must be the first of the non-cutting lines. It will develop that these two lines parallel to AB have properties quite different from the other lines through P which do not cut AB.

The results which followed as a consequence of these assumptions aroused the greatest wonder in the young Bolyai. As the geometry developed and no contradictions appeared, this wonder grew and he began to feel something of the significance of what he was doing. What seemed to impress him most were the propositions which did not depend upon any parallel postulate at all, but which were common to all geometries regardless of what assumptions were made about parallels. These he regarded as stating absolute facts about space and forming the basis of an absolute geometry.

These ideas had certainly begun to take form, however vaguely, by 1813 when Bolyai was only twenty-one years old. The following extract from a letter[10], written to his father on November 3, 1813, shows how far he had gone with his discoveries and how deeply he was affected by them.

"It is now my definite plan to publish a work on parallels as soon as I can complete and arrange the material and an opportunity presents itself; at the moment I still do not clearly see my way through, but the path which I have followed gives positive evidence that the goal will be reached, if it is at all possible; I have not quite reached it, but I have discovered such wonderful things that I was amazed and it would be an everlasting piece of bad fortune if they were lost. When you, my dear Father, see them, you will understand; at present I can say nothing except this: that *out of nothing I have created a strange new universe.* All that

I have sent you previously is like a house of cards in comparison with a tower. I am no less convinced that these discoveries will bring me honor, than I would be if they were completed."

In reply, the elder Bolyai suggested that the proposed work be published as an appendix to his *Tentamen*, and urged that this be done with as little delay as possible.[11] But the formulation of results and the expansion of ideas came slowly. In February, 1815, however, Johann visited his father and brought along an outline of his work. Finally in 1819 he submitted his manuscript and, despite the fact that father and son disagreed on a few points, there was published in 1832 the *Appendix*.[12]

Previously, in 1831, eager to know what Gauss would have to say about his son's discoveries, Wolfgang had sent him an abridgment of the *Appendix*, but it failed to reach him. In February, 1831, Gauss received an advance copy of the *Appendix*. His response,[13] written to Wolfgang on March 6, 1831, contains the following remarks about the work of Johann.

"If I begin with the statement that I dare not praise such a work, you will of course be startled for a moment: but I cannot do otherwise; to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. On this account I find myself surprised to the extreme.

"My intention was, in regard to my own work, of which very little up to the present has been published, not to allow it to become known during my lifetime. Most people have not the insight to understand our conclusions and I have encountered only a few who received with any particular interest what I communicated to them. In order to understand these things, one must first have a keen perception of what is needed, and upon this point the majority are quite confused. On the other hand it was my plan to put all down on paper eventually, so that at least it would not finally perish with me.

"So I am greatly surprised to be spared this effort, and am overjoyed that it happens to be the son of my old friend who outstrips me in such a remarkable way."

When Johann received a copy of this letter from his father he was far from elated. Instead of the eulogies which he had anticipated, it brought him, in his opinion, only the news that another had made the same discoveries independently and possibly earlier. He even went so far as to suspect that, before the *Appendix* was completed, his father had confided some of his ideas to Gauss, who in turn had appropriated them for his own use. These suspicions were eventually dispelled, but Johann never felt that Gauss had accorded him the honor that was his due.

Johann Bolyai published nothing more, though he continued his investigations. Notes found among his papers show that he was interested in the further extension of his ideas into space of three dimensions and also in the comparison of his Non-Euclidean Geometry with Spherical Trigonometry. It was this latter comparison which led him to the conviction that the Fifth Postulate could not be proved.[14] He was never thoroughly convinced, however, that investigations into space of three dimensions might not lead to the discovery of inconsistencies in the new geometry.

In 1848 Bolyai learned that the honor for the discovery of Non-Euclidean Geometry must be shared with still another. In that year he received information of Lobachewsky's discoveries and examined them critically. There was aroused in him the spirit of rivalry, and in an attempt to outshine Lobachewsky he began to labor in earnest again on what was to be his great work, the *Raumlehre*, which he had planned when he was publishing the *Appendix*. But this work was never completed.

31. Lobachewsky.

Although it was not until 1848 that Bolyai learned of the work of Nikolai Ivanovich Lobachewsky (1793–1856), the latter had discovered the new geometry and had actually published his conclusions as early as 1829, two or three years before the appearance in print of the *Appendix*. But there is ample evidence that he made his discoveries later than Bolyai made his.

Lobachewsky[15] took his degree at the University of Kasan in 1813. He was retained as instructor and later was promoted to a professorship. As a student there he had studied under Johann M. C. Bartels, who had been one of the first to recognize the genius of Gauss. Although Gauss and Bartels were close friends, there is no evidence that the latter, when he went to Kasan in 1807, carried with him and imparted to Lobachewsky any advanced views on the problem of parallels. Indeed, we know that Gauss himself at that early date was still working along conventional lines. The later discoveries of Lobachewsky seem to have been the results of his own initiative, insight and ability.

At any rate, along with the others, he was trying to prove the Fifth Postulate as early or as late as 1815. A copy of the lecture notes, taken by one of his students during that year and the two following, reveals only attempts to verify the Euclidean theory. It was not until after 1823 that he began to change his viewpoint, by which date, it will be recalled, Johann Bolyai had reached pretty well organized ideas about his new geometry.

In 1823 Lobachewsky had completed the manuscript for a textbook on elementary geometry, a text which was never published. This manuscript is extant. In it he made the significant statement that no rigorous proof of the Parallel Postulate had ever been discovered and that those proofs which had been suggested were merely explanations and were not mathematical proofs in the true sense. Evidently he had begun to realize that the difficulties encountered in the attempts to prove the Postulate arose through causes quite different from those to which they had previously always been ascribed.

The next three years saw the evolution of his new theory of parallels. It is known that in 1826 he read a paper before the physics and mathematics section of the University of Kasan and on that occasion suggested a new geometry in which more than one straight line can be drawn through a point parallel to a given line and the sum of the angles of a triangle is less than two right angles. Unfortunately the lecture was never printed and the manuscript has not been found.

But in 1829–30 he published a memoir on the principles of geometry in the *Kasan Bulletin*, referring to the lecture mentioned above, and explaining in full his doctrine of parallels. This memoir, the first account of Non-Euclidean Geometry to appear in print, attracted little attention in his own country, and, because it was printed in Russian, practically none at all outside.

Confident of the merit of his discoveries, Lobachewsky wrote a number of papers, more or less extensive, on the new theory of parallels, hoping thus to bring it to the attention of mathematicians all over the world. Perhaps the most important of these later publications was a little book entitled *Geometrische Untersuchungen zur Theorie der Parallellinien*,[16] written in German with the idea that it might for that reason be more widely read. A year before his death, although he had become blind, he wrote a complete account of his researches which was published in French under the title: *Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèls.*[17] But he did not live to see his work accorded any wide recognition.

So slowly was information of new discoveries circulated in those days that Gauss himself did not learn of the advances made by Lobachewsky for a number of years, perhaps not until after the publication of the *Untersuchungen*. At any rate, it appears that by 1841 he knew of Lobachewsky and his work and was deeply impressed. In 1846 he wrote to Schumacher as follows:[18]

"I have recently had occasion to look through again that little volume by Lobatschefski (Geometrische Untersuchungen zur Theorie der Parallellinien, Berlin 1840, bei B. Funcke, 4 Bogen stark). It contains the elements of that geometry which must hold, and can with strict consistency hold, if the Euclidean is not true. A certain Schweikardt[19] calls such geometry Astral Geometry, Lobatschefsky calls it Imaginary Geometry. You know that for fifty-four years now (since 1792.) I have held the same conviction (with a certain later extension, which I will not mention here). I have found in Lobatschefsky's work nothing that is new to me, but the development is made in a way different from that which I have followed, and certainly by Lobatschefsky in a skillful way and in truly geometrical spirit. I feel that I must call your attention to the book, which will quite certainly afford you the keenest pleasure."

By 1848 Wolfgang Bolyai had heard in some way of Lobachewsky's investigations. In January of that year he wrote to Gauss, asking for the name of the book by the Russian mathematician. Gauss recommended "that admirable little work," the *Geometrische Untersuchungen*, as containing an adequate exposition of the theory and as being easily obtainable. Thus Wolfgang and, through him, Johann became acquainted with the geometry of Lobachewsky.

That Johann received this information about the work of the Russian geometer philosophically enough is evinced by remarks found in his unpublished notes entitled: *Bemerkungen über Nicolaus Lobatchefskij's Geometrische Untersuchungen*. He wrote in part:[20]

"Even if in this remarkable work different methods are followed at times, nevertheless, the spirit and result are so much like those of the *Appendix* to the *Tentamen matheseos* which appeared in the year 1832 in Maros-Vásárhely, that one cannot recognize it without wonder. If Gauss was, as he says, surprised to the extreme, first by the *Appendix* and later by the striking agreement of the Hungarian and Russian mathematicians: truly, none the less so am I.

"The nature of real truth of course cannot but be one and the same in Maros-Vásárhely as in Kamschatka and on the Moon, or, to be brief, anywhere in the world; and what one finite, sensible being discovers, can also not impossibly be discovered by another."

But, regardless of these reflections, for a time at least, Bolyai entertained the suspicion that somehow Lobachewsky had learned of his own discoveries, possibly through Gauss, and had then, after some revision, published them. His attitude later, however, became somewhat more lenient. As a matter of fact, there seems to be no evidence that Lobachewsky ever heard of Bolyai.

32. Wachter, Schweikart and Taurinus.

No satisfactory record, however brief, of the discovery of Non-Euclidean Geometry will fail to include the names of Wachter, Schweikart, and Taurinus. We insert here short accounts of their contributions, before turning our attention to further developments due to Riemann and others.

Friedrich Ludwig Wachter (1792–1817), Professor of Mathematics in the Gymnasium at Dantzig, studied under Gauss at Göttingen in 1809. His attempts to prove the Fifth Postulate led to the publication in 1817 of a paper[21] in which he attempted to prove that through any four points in space, not lying in one plane, a sphere can be constructed. This plan of investigation was obviously suggested by the fact that the Postulate can be proved once it is established that a circle can be drawn through any three non-collinear points. Although his arguments were unsound, some of his intuitive deductions in this paper, and in a letter[22] written to Gauss in 1816, are worthy of recognition. Among other things, he remarked that, even if Euclid's Postulate is denied, spherical geometry will become Euclidean if the radius of the sphere is allowed to become infinite, although the limiting surface *is not a plane.* This was confirmed later by both Bolyai and Lobachewsky.

Wachter lived only twenty-five years. His brief investigations held much promise and exhibited keen insight. Had he lived a few years longer he might have become the discoverer of Non-Euclidean Geometry. As it was, his influence was probably considerable. Just at the time when he and Gauss were discussing what they called *Anti-Euclidean Geometry*, the latter began to show signs of a change of viewpoint. In 1817, writing to H. W. M. Olbers, his associate and a noted astronomer, Gauss was led to remark, after mentioning Wachter, and commending his work despite its imperfections,[23] "I keep coming closer to the conviction that the necessary truth of our geometry cannot be proved, at least *by* the human intellect *for* the human intellect. Perhaps in another life we shall arrive at other insights into the nature of space which at present we cannot reach. Until then we must place geometry on an equal basis, not with arithmetic, which has a purely *a priori* foundation, but with mechanics."

It will be recalled[24] that Gauss, in a letter to Schumacher, mentioned "a certain Schweikardt." The one referred to was Ferdinand Karl Schweikart (1780–1859), who from 1796 to 1798 was a student of law at Marburg. As he was keenly interested in mathematics, he took advantage of the opportunity while at the university to listen to the lectures of J. K. F. Hauff, who was somewhat of an authority on the theory of parallels. Schweikart's interest in this theory developed to such an extent that in 1807 there appeared his only published work of a mathematical nature, *Die Theorie der Parallellinien nebst dem Vorschlage ihrer Verbannung aus der Geometrie*.[25] In spite of its title, this book offered nothing particularly novel and was written along quite conventional lines. In it he mentioned both Saccheri and Lambert. His acquaintance with the work of these men doubtless affected the character of his later investigations. In 1812 Schweikart went to Charkow; the year 1816 found him in Marburg again, where he remained until 1820 when he became Professor of Jurisprudence at Königsberg.

In 1818 he handed to his friend Gerling, student of Gauss and Professor of Astronomy at Marburg, a brief outline of his ideas about a new geometry in which the Parallel Postulate was denied, and asked him to forward it to Gauss for his criticism. In this memorandum he asserted that there are two kinds of geometry, Euclidean and *Astral* and that in the latter the sum of the angles of a triangle is less than two right angles; the smaller the angle-sum, the greater the area of the triangle; that the altitude of an isosceles right triangle increases as the sides increase, but can never become greater than a certain length called the *Constant*; that, when this Constant is taken as infinite, Euclidean Geometry results. This outline was probably the first explicit description of a Non-Euclidean Geometry, regarded as such. The ideas came to Schweikart before 1816, while he was still in Charkow. At that early date both Bolyai and Lobachewsky were still carrying on their investigations from the traditional viewpoint.

In his reply to Gerling, Gauss commended Schweikart highly. "The memorandum of Prof. Schweikardt has brought me the greatest pleasure," he wrote, "and in regard to it please extend to him my sincerest compliments. It might almost have been written by myself."

Schweikart did not publish the results of any of his investigations. But he did encourage his sister's son, Franz Adolph Taurinus (1794–1874), to take up the study of parallels, suggesting that he give some thought to the Astral Geometry which Gauss had praised so highly. Taurinus, after studying jurisprudence for a brief time, had settled down in Köln to spend a long life of leisure, with ample time to devote to varied intellectual interests. In 1824, when he first began a systematic investigation of the problem of parallels, he found himself not in accord with his uncle's ideas. That he hoped, at this early point in his researches, to be able to prove the Fifth Postulate is nothing out of the ordinary. The remarkable fact is that, although as a consequence of his independent investigations he was one of the first to obtain a view of Non-Euclidean Geometry, nevertheless throughout his life he continued to believe that the Euclidean Hypothesis was the only one of the three which would lead to a valid geometry.

In 1825, soon after he had received from Gauss the complimentary and encouraging letter which was translated in full in Section 29, appeared his first book, *Theorie der Parallellinien*.[26] Here he attacked the problem from the Non-Euclidean viewpoint, rejecting the Hypothesis of the Obtuse Angle and, using the Hypothesis of the Acute Angle, encountered the Constant of Schweikart. These investigations led him to ideas which were not in accord with his concept of space and he was impelled to reject the latter hypothesis also, although he appeared to recognize the consequences as logically sound.

Shortly after the publication of his first book he learned that Saccheri and Lambert had both preceded him along the route he had followed. So he produced another book in 1826, his *Geometriae Prima Elementa*,[27] in which he modified his method of attack. It was in the appendix of this work that he made his most important contributions. Here he developed many of the basic formulas for Non-Euclidean Trigonometry. In the familiar formulas of spherical trigonometry he replaced the real radius of the sphere by an imaginary one. The modified formulas, remarkably enough, describe the geometry which arises under the Hypothesis of the Acute Angle. Lambert had previously investigated trigonometric functions with imaginary arguments, and in that connection had developed to some extent the theory of hyperbolic functions, but there is no evidence that he tried to use these ideas in his study of parallels. It will be recalled that he had surmised that this geometry might be verified on a sphere of imaginary radius.[28] Taurinus did not use hyperbolic functions; instead he exhibited the real character of his formulas through the medium of exponents and logarithms. Consequently he called the geometry *Logarithmisch-Sphärischen Geometrie*. He, as had Lambert, recognized the correspondence between spherical geometry and that which arises if the Hypothesis of the Obtuse Angle is used. In addition, he noted that his Logarithmic-Spherical Geometry became Euclidean when the radius of the sphere was made infinite.

Although his reluctance to recognize this geometry as valid on a plane persisted, Taurinus seemed to be fully aware of the importance of his discoveries, from the theoretical viewpoint, in the study of parallels. His *Geometriae Prima Elementa* received little recognition. In his disappointment, he burned the remaining copies.

33. Riemann.

Neither Bolyai nor Lobachewsky lived to see his work accorded the recognition which it merited. This delay can be attributed to several factors: the slow passage of ideas from one part of the world to another, the language barriers, the Kantian space philosophy, the two-thousand-year dominance of Euclid, and the relative obscurity of the discoverers of Non-Euclidean Geometry. The new geometry attracted little attention for over thirty-five years until, in 1867, Richard Baltzer, in the second edition of his *Elemente der Mathematik*, inserted a reference to it and its discoverers, and also persuaded Hoüel to translate their writings into French.

But in the meantime a new figure had appeared. Born at about the time of the discovery of Non-Euclidean Geometry, George Friedrich Bernhard Riemann (1826–1866) grew to young manhood with the intention of studying theology. But when he entered Göttingen for that purpose he discovered that mathematics was his forte and gave up theology. He studied under Gauss and became the outstanding student in the long teaching career of that great mathematician. Later he went to Berlin to study with Dirichlet, Jacobi, Steiner, and others, but returned to Göttingen in 1850 to study physics and take his degree there the following year.

We have already quoted[29] from the remarkable probationary lecture, *Über die Hypothesen welche der Geometrie zu Grunde liegen*, which he delivered in 1854 before the Philosophical Faculty at Göttingen, and in which he pointed out that space need not be infinite, though regarded as unbounded. Thus he suggested indirectly a geometry in which no two lines are parallel and the sum of the angles of a triangle is greater than two right angles. It will be recalled that, in the rejection of the Hypothesis of the Obtuse Angle by earlier investigators, the infinitude of the line had been assumed.

But Riemann, in this memorable dissertation, did more than that; he called attention to the true nature and significance of geometry and did much to free mathematics of the handicaps of tradition. Among other things, he said,[30] "I have in the first place .... set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. But hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure relations of space — the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are — like all matters of fact — not necessary but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small."

Stressing the importance of the study of the properties of things from the infinitesimal standpoint, he continued, "The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends. The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo, and Newton, and used by modern physics. But in the natural sciences which are still in want of simple principles for such constructions, we seek to discover the causal relations by following the phenomena into great minuteness, so far as the microscope permits. Questions about the measure-relations of space in the infinitely small are not therefore superfluous questions."

Thus began a second period in the development of Non-Euclidean Geometry, a period characterized by investigations from the viewpoint of differential geometry in contrast with the synthetic methods previously used. Riemann's memoir dealt almost altogether with generalities and was suggestive in nature. The detailed investigations along these lines were carried out by others, notably Helmholtz, Lie, and Beltrami. The contributions of the physicist, Helmholtz, remarkable as they were, required for rigor the finishing touches of a mathematician. These thorough investigations were made by Lie, using the idea of groups of transformations. To Beltrami goes the credit of offering the first proof of the consistency of Non-Euclidean Geometry. Although Bolyai and Lobachewsky had encountered no contradiction in their geometry as far as their investigations had gone, there still remained the possibility that some such inconsistency might arise as the research continued. Beltrami showed how this geometry can be represented, with restrictions, on a Euclidean surface of constant curvature, and thus how any inconsistency discovered in the geometry of Bolyai and Lobachewsky will lead to a corresponding one in Euclidean Geometry.

34. Further Developments.

The work of this second period was excellent and the results were far reaching and significant, but it remained for a third period — one with which are associated the names of Cayley, Klein, and Clifford — to supply what was still needed in the way of the unification and interpretation of the Non-Euclidean Geometries. The beautiful classification of these geometries from the projective metric viewpoint and the recognition of the roles which they play in the rounding out of a logical category led to their complete justification and thus brought to a triumphant close the long struggle with the Fifth Postulate.

In his notable *Sixth Memoir upon Quantics*,[31] Cayley, in 1859, showed how the notion of distance can be established upon purely descriptive principles. These ideas were developed and interpreted from the standpoint of Non-Euclidean Geometry by Felix Klein in two monographs[32] appearing in 1871 and 1873. It was he who suggested calling the geometries of Bolyai and Lobachewsky, Riemann, and Euclid, respectively, *Hyperbolic, Elliptic, and Parabolic*, a terminology almost universally accepted and which we shall use from this point on. The names were suggested by the fact that a straight line contains two infinitely distant points under the Hypothesis of the Acute Angle, none under the Hypothesis of the Obtuse Angle, and only one under the Hypothesis of the Right Angle. More recently investigators have confined their attentions largely to careful scrutiny of the foundations of geometry and to the precise formulation of the sets of axioms. Following the lead of Pasch, such men as Hilbert, Peano, Pieri, Russell, Whitehead, and Veblen have gone far in placing geometry, both Euclidean and Non-Euclidean, as well as mathematics in general, on a firm logical basis.

35. Conclusion.

In the following pages we shall take up first a study of Synthetic Hyperbolic Geometry. This will be followed by an investigation of the trigonometry of the Hyperbolic Plane and that, in turn, by a brief treatment from the viewpoint of analytic geometry and calculus.

Our examination of Elliptic Geometry will be less extensive. Its development, like that of most of the later work in Non-Euclidean Geometry, depends upon the use of concepts more advanced than those which we wish to draw upon here.

In passing, we remark that there are two types of Elliptic Geometry. The one suggested in Section 6 is probably the one which Riemann had in mind. Geometry on this Elliptic Plane has an exact analogue in the geometry on a sphere, the great circles being regarded as straight lines. The other type, in many respects the more interesting and important, was suggested later by Klein. In this geometry, two points always determine a straight line, and in other respects it more nearly resembles Euclidean Geometry

Footnotes

1. Paul Stäckel: *Wolfgang und Johann Bolyai, Geometrische Untersuchungen*, Vol. I, p. 86 (Leipzig and Berlin, 1913).
2. Engel and Stäckel have collected some of these in their sourcebook, *Die Theorie der Parallellinien von Euklid bis auf Gauss* (Leipzig, 1895).
3. For a photographic facsimile of this letter, see Engel and Stäckel, *loc. cit.*
4. See Engel and Stäckel, *loc. cit.*, p. 230.
5. Pronounced Bol′yah-eh.
6. Engel and Stäckel, *loc. cit.*, p. 219.
7. See Stäckel, *Wolfgang und Johann Bolyai*, Vol. II, p. 5, for a German translation. The original was in Latin.
8. Stäckel, *loc. cit.*, Vol. II, p 16.
9. The title is: *Tentamen juventutem studiosam in elementa matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentiaque huic propria, introducendi. Cum appendice triplice.* See Stäckel, *loc. cit.*, Vol. II, p. 25.
10. Stäckel, *loc. cit.*, Vol. I, p. 85.
11. See Section 28.
12. The complete title is: *Appendix Scientiam Spatii absolute veram exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud umquam decidenda) independentem: adjecta ad casum falsitatis, quadratura circuli geometrica*. See the German translation by Stäckel, *loc. cit.*, Vol. II, p. 183, the English translation by G. B. Halsted, 4th edition (Austin, Texas, 1896), or David Eugene Smith: *A Source Book in Mathematics*, p. 375 (New York, 1929).
13. Stäckel, *loc. cit.*, Vol. I, p. 92.
14. Stäckel, *loc. cit.*, Vol I, p. 121.
15. Perhaps the best account of Lobachewsky and his work is to be found in Friedrich Engel: *N. I. Lobatschefskij*, (Leipzig, 1898).
16. See Lobatschewsky: *Geometrical Researches on the Theory of Parallels*, translated by G. B. Halsted (Austin, Texas, 1891).
17. See David Eugene Smith: *A Source Book in Mathematics*, p. 360 (New York, 1929).
18. Engel and Stäckel: *Die Theorie der Parallellinien von Euklid bis auf Gauss*, p. 235.
19. See Section 32.
20. Stäckel: *Wolfgang und Johann Bolyai, Geometriche Untersuchungen*, Vol. I, p. 140 (Leipzig and Berlin, 1913).
21. For this paper and some of Wachter's letters, see Stäckel: *Friedrich Ludwig Wachter, ein Beitrag zur Geschichte der Nichteuklidischen Geometrie*, Mathematische Annalen, Vol. 54, pp. 49–85 (1901).
22. Stäckel, *ibid.*, p. 61.
23. Stäckel, *ibid.*, p. 55.
24. See Section 31.
25. See Engel and Stäckel, *loc. cit.*, p. 243.
26. For an excerpt, see Engel and Stäckel, *loc. cit.*, p. 255.
27. Engel and Stäckel, *loc. cit.*, p. 267.
28. See Section 23.
29. See Section 6.
30. Clifford's translation, *Nature*, Vol. VIII, 1873. See also David Eugene Smith, *A Source Book tn Mathematics*, pp. 411–425 (New York, 1929).
31. See Cayley's *Collected Mathematical Papers*, Vol. II, pp. 561–592 (Cambridge, 1889).
32. See his *Gesammelte Mathematische Abhandlungen*, Vol. I, pp. 254–305 and pp. 311–343 (Berlin, 1921).