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Graph Puzzles, Homotopy, and the Alternating Group*

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The so-called 15-puzzle may be generalized to a puzzle based on an arbitrary graph. We consider labelings or colorings of the vertices and the operation of switching one distinguished label with a label on an adjacent vertex. Starting from a given labeling, iterations of this operation allow one to obtain all, or exactly half, of the labelings on a non-separable graph (with the polygons and one other graph as exceptions).

1. THE PUZZLE

The well known "15-puzzle" consists of fifteen small movable square tiles numbered 1, 2,..., 15 and one empty square, arranged in a 4×4 array. One is permitted to interchange the empty square with a tile next to it as often as desired. The challenge is to move by a sequence of such interchanges from one given position of the tiles to another specified position. This may or may not be possible. It was observed as early as 1879 [3] that the existence of a solution to the problem depends on the parity of the permutation of the squares required to map the first position onto the second.

A discussion of the 15-puzzle containing this observation may also be found in Chapter 1 of [2]. This exposition motivated the author to consider the analogous puzzle based on an arbitrary simple graph (i.e., a graph without loops or "multiple" edges). However, the problem has been raised independently by R. Stanley. D. Greenwell and L. Lovász also came upon this problem independently and have obtained proofs of Theorems 1 and 2 below for non-bipartite graphs with at least nine vertices.

Let G be a finite simple graph with vertex set $V(G)$ of cardinality $n + 1$. By a labeling we mean the placement of labels 1, 2,..., n on distinct vertices

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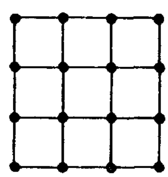
of G, leaving one vertex "blank" or "unoccupied." Formally, a *labeling* on G is to be bijective mapping $f\colon V(G) \to \{1, 2,..., n, \varnothing\}$. The vertex x with $xf = \varnothing$ is said to be *unoccupied* in f.

(All mappings will be written on the right and composition of mappings is to be read from left to right.)

Two labelings f, g on G are said to be adjacent if and only if g can be obtained from f by "sliding" a label along an edge of G onto the vertex which is unoccupied in f. Formally, labelings f, g on G are *adjacent* if and only if $g = (xy) \cdot f$, where (xy) denotes the transposition (permutation of $V(G)$) which interchanges x and y, x is the vertex with $xf = \varnothing$, and y is any vertex of G adjacent to x in G. Thus $yg = \varnothing$ and $xg = yf$. (This operation of switching $\varnothing$ with the label on an adjacent vertex arises in the recoloring procedure in the standard proof of Brook's Theorem.)

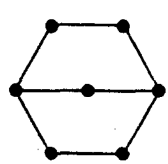
The relation of adjacency defined on the labelings is symmetric and irreflexive, and so defines a new simple graph puz(G) with vertex set $V(\text{puz}(G))$ consisting of all labelings on G, two labelings being joined by an edge in puz(G) if and only if they are adjacent. The "puzzle" referred to in the title consists of determining whether two given labelings f, g are in the same connected component of puz(G), and when possible, explicitly constructing a path from f to g in puz(G).

The classic 15-puzzle is of this type. Here the graph G is:



In this case, puz(G) has exactly two components, each containing $\frac{1}{2}(16!)$ labelings.

An interesting graph turns out to be the graph θ_0 defined by the diagram:



THEOREM 1. *Let G be a finite simple non-separable graph other than a polygon or the graph θ_0 above. Then* puz(G) *is connected unless G is bipartite, in which case* puz(G) *has exactly two components. In this latter case, labelings f, g on G having unoccupied vertices at even (respectively, odd) distance in G are in the same component of* puz(G) *if and only if fg^{-1} is an*

even (respectively, odd) permutation of $V(G)$. $\text{puz}(\theta_0)$ *has exactly six components.*

2. Paths and Permutations

A *path* p in a graph G is, for our purposes, a sequence

$$p = (x_0, x_1, ..., x_n)$$

of vertices of G such that x_{i-1} and x_i are adjacent in G, $i = 1, 2, ..., n$. Such a path p is said to be *from* x_0 (its *initial* vertex) *to* x_n (its *terminal* vertex). The path p is *simple* when $x_0, x_1, ..., x_n$ are distinct, with the possible exception that $x_0 = x_n$, in which case p is a *simple closed* path. The path $\bar{p} = (x_n, ..., x_1, x_0)$ is the *reverse* of p.

For each path $p = (x_0, x_1, ..., x_n)$ in G, define a permutation σ_p of $V(G)$ as the product of transpositions

$$\sigma_p = (x_0x_1)(x_1x_2)(x_2x_3) \cdots (x_{n-1}x_n).$$

The following proposition is immediate from the definitions.

Proposition 1. *Labelings f, g on a graph G are in the same component of* $\text{puz}(G)$ *if and only if $f = \sigma_p g$ for some path p in G from $f^{-1}(\varnothing)$ to $g^{-1}(\varnothing)$.*

The path p mentioned in Proposition 1 is the path followed by the unoccupied vertex in deriving g from f.

Define $\Gamma(x, y) = \Gamma_G(x, y)$ to be the set of all permutations σ_p of $V(G)$ where p is a path from x to y in G. We abbreviate $\Gamma(x, x) = \Gamma_G(x, x)$ by $\Gamma(x) = \Gamma_G(x)$.

If p is a path from x to y and q a path from y to z, then the product pq, a path from x to z, is defined as usual. Note that $\sigma_p\sigma_q = \sigma_{pq}$, and $\sigma_p^{1} = \sigma_{\bar{p}}$. These observations lead to

Proposition 2. *For each vertex x of G, $\Gamma(x)$ is a group of permutations of $V(G)$, each fixing x. If p is a path from x to y in G, then $\Gamma(x, y) = \Gamma(x)\,\sigma_p = \sigma_p\Gamma(y)$ and $\Gamma(y) = \sigma_p^{-1}\Gamma(x)\,\sigma_p$.*

Remarks. It is perhaps relevant to observe that, if p and q are homotopic paths, then $\sigma_p = \sigma_q$. (Paths p, q in a simple graph are homotopic when one can be derived from the other by a sequence of replacements of vertex terms y by paths (y, z, y), and the inverse of this operation.) Thus $p \mapsto \sigma_p$ induces a homomorphism of the fundamental group of G based at x onto $\Gamma(x)$. The fundamental group of a polygon with n vertices is generated by a simple closed path $p = (x_0, x_1, ..., x_{n-1}, x_0)$ and here $\sigma_p = (x_{n-1} \cdots x_2 x_1)$, an $(n-1)$-cycle. Hence $\Gamma(x_0)$ is cyclic of order $n-1$ in this case. The reader may verify that $\Gamma(x)$ is transitive on $V(G) - \{x\}$ if and only if G is non-separable.

THEOREM 2. *Let G be a finite simple non-separable graph other than a polygon or the graph θ_0. Then, for any vertex x of G,*

$$\Gamma(x) = \mathrm{sym}(V(G) - \{x\}),$$

unless G is bipartite, in which case

$$\Gamma(x) = \mathrm{alt}(V(G) - \{x\}).$$

If $G = \theta_0$, then, for each vertex x, $\Gamma(x)$ is the group $\mathrm{PGL}_2(5)$ *of order* 120 *in its sharply* 3*-transitive representation of degree* 6.

Here, for any set X, $\mathrm{sym}(X)$ denotes the symmetric group of all permutations of X and $\mathrm{alt}(X)$ denotes the alternating group of all even permutations of X.

In the statement of Theorem 2, we make the minor logical error of not distinguishing between permutations of $V(G) - \{x\}$ and permutations of $V(G)$ which fix x. This convention is continued throughout this paper.

Theorem 1 follows quickly from Theorem 2 and Propositions 1 and 2:

Proof of Theorem 1. Let G be a connected graph and fix $x \in V(G)$. Each component of $\mathrm{puz}(G)$ contains a labeling f with $xf = \varnothing$. Labelings f, g with $xf = xg = \varnothing$ are in the same component of $\mathrm{puz}(G)$ if and only if $fg^{-1} \in \Gamma(x)$. Thus the components of $\mathrm{puz}(G)$ are in one-to-one correspondence with the right cosets of $\Gamma(x)$ in $\mathrm{sym}(V(G) - \{x\})$, and their number is the index of $\Gamma(x)$ in $\mathrm{sym}(V(G) - \{x\})$.

Suppose now that $\Gamma(x) = \mathrm{alt}(V(G) - \{x\})$ for all vertices x of a connected graph G. Let f, g be labelings on G with $xf = yg = \varnothing$ and choose a path p from x to y. Then f and g are in the same component of $\mathrm{puz}(G)$ if and only if $fg^{-1} \in \Gamma(x, y) = \Gamma(x)\,\sigma_p$ if and only if fg^{-1} and σ_p have the same parity if and only if fg^{-1} and the length of p have the same parity.

Our proof of Theorem 2 proceeds by induction on the cyclomatic number (or Betti number) $\beta(G) = |E(G)| - |V(G)| + 1$ of the non-separable graph G. Non-separable graphs with $\beta(G) = 1$ are the polygons; non-separable graphs with $\beta(G) = 2$ are the "θ-graphs," of which θ_0 is a representative. The proof is given in Section 4 following a section containing lemmas required to start the induction with the θ-graphs, and to complete the proof.

We remark at this point that, for the proof of Theorem 2, it will suffice to verify the assertion for a single vertex x, since Proposition 2 shows that, for a connected graph G, the groups $\Gamma(x)$, $x \in V(G)$, are equivalent permutation groups. Also, it will be enough to show that

$$\mathrm{alt}(V(G) - \{x\}) \subset \Gamma(x)$$

for graphs G satisfying the hypothesis. For, if G is connected, there will be closed paths p based at x of odd length (or equivalently, odd permutations σ_p in $\Gamma(x)$) if and only if G is not bipartite.

3. Generating the Alternating Group

LEMMA 1. *Let X be a finite set, $|X| \geqslant 3$, and fix $u, v \in X$. Then the 3-cycles (uvx), $x \in X - \{u, v\}$, generate* alt (X).

Proof. A proof may be found in Chapter 1 of [1], but it is short enough to sketch here. Each $\sigma \in \mathrm{alt}(x)$ can be written as a product of an even number of transpositions. Since $(xy) = (vx)(vy)(vx)$, we can write $\sigma = (vx_1)(vx_2) \cdots (vx_m)$ where m is even. Then

$$\sigma = (vx_1x_2)(vx_3x_4) \cdots (vx_{m-1}x_m).$$

If $x_{2i} \neq u$, replace $(vx_{2i-1}x_{2i})$ by the product $(uvx_{2i-1})^{-1}(uvx_{2i})(uvx_{2i-1})$ in this latter expression for σ.

For $\sigma \in \mathrm{sym}(X)$, define the *support* $\|\sigma\|$ of σ to be the set of all elements $x \in X$ which are *not* fixed by σ. Thus $\|\sigma\|$ contains two elements when σ is a 2-cycle (transposition) and $\|\sigma\|$ contains three elements when (and only when) σ is a 3-cycle.

LEMMA 2. *Let Σ be a set of 3-cycles on a finite set X, $|X| \geqslant 3$, and let $\langle\Sigma\rangle$ denote the subgroup of* sym(X) *generated by Σ. Then the following are equivalent*:

(i) $\langle\Sigma\rangle = \mathrm{alt}(X)$.

(ii) *$\langle\Sigma\rangle$ is transitive on X.*

Proof. For a subgroup $\Gamma \subseteq \mathrm{sym}(X)$ and a subset $Y \subseteq X$, we define $\Gamma \mid Y = \{\sigma \in \Gamma : \|\sigma\| \subseteq Y\}$, considered as a subgroup of sym(Y).

It is clear that (i) implies (ii). Assume Σ is given satisfying (ii). Then let Y be a subset of X such that $|Y| \geqslant 3$, $\langle\Sigma\rangle \mid Y = \mathrm{alt}(Y)$, and which is maximal with respect to these properties. We claim $Y = X$ (which will prove the lemma).

If $Y \neq X$, our assumption (ii) implies that either: (a) there exists a 3-cycle $(uvz) \in \Sigma$ with $u, v \in Y, z \notin Y$; or, (b) there exists a 3-cycle $(uvz) \in \Sigma$ with $z \in Y, u, v \notin Y$. In case (a), $\langle \Sigma \rangle$ contains (uvx), $x \in Y - \{u, v\}$, in addition to (uvz), so $\langle \Sigma \rangle | (Y \cup \{z\}) = \text{alt}(Y \cup \{z\})$ by Lemma 1. This contradicts the maximality of Y. In case (b), for each $x \in Y$, $\langle \Sigma \rangle | Y$ contains a permutation σ such that $z\sigma = x$. Then $\sigma^{-1}(uvz)\,\sigma = (uvx) \in \langle \Sigma \rangle$. This holds for every $x \in Y$ and hence $\langle \Sigma \rangle | (Y \cup \{u, v\}) = \text{alt}(Y \cup \{u, v\})$ by Lemma 1. Again, the maximality of Y is contradicted.

LEMMA 3. *Let Γ be a transitive permutation group on X and suppose that Γ contains a 3-cycle. If Γ is primitive (in particular, if Γ is doubly transitive), then* $\text{alt}(X) \subseteq \Gamma$.

Proof. Let Σ be the set of 3-cycles in Γ. Then the orbits of $\langle \Sigma \rangle$ on X are clearly sets of imprimitivity for Γ. Hence, if Γ is primitive, then $\langle \Sigma \rangle$ is transitive and Lemma 2 completes the proof.

LEMMA 4. *Let $X = \{y, a_1, a_2, \ldots, a_n, z, b_1, b_2, \ldots, b_m\}$ be a set of $n + m + 2$ letters ($n \geqslant m \geqslant 0$) and let*

$$\pi = (a_1 a_2 \cdots a_n z b_1 b_2 \cdots b_m),$$

$$\rho = (b_1 b_2 \cdots b_m y a_1 a_2 \cdots a_n).$$

Then $\langle \pi, \rho \rangle = \text{sym}(X)$ *if $n + m$ is odd, and* $\langle \pi, \rho \rangle = \text{alt}(X)$ *if $n + m$ is even, unless*

(i) $n = 2, m = 1$,

(ii) $n = m = 2$, *or*

(iii) $n = 4, m = 2$.

(In case (i), $\langle \pi, \rho \rangle$ is of order 20; in case (ii), $\langle \pi, \rho \rangle$ is of order 60; in case (iii), $\langle \pi, \rho \rangle$ is of order 1344.)

Proof. To prove our assertion (assuming (i), (ii), (iii) do not hold), it will suffice to show that $\text{alt}(X) \subseteq \langle \pi, \rho \rangle$, since π and ρ have the same parity as $m + n$.

The group $\langle \pi, \rho \rangle$ is doubly transitive on X since it is obviously transitive and the stabilizer of, say, z in $\langle \pi, \rho \rangle$ contains ρ and so is transitive on $X - \{z\}$. Thus, in view of Lemma 3, it will be enough to show that $\langle \pi, \rho \rangle$ contains a 3-cycle (except in cases (i), (ii), and (iii)).

First assume $n \geqslant m \geqslant 3$. Define permutations $\sigma_1, \sigma_2, \ldots, \sigma_9 \in \langle \pi, \rho \rangle$ as follows:

$$\begin{aligned}
\sigma_1 &= \pi^{-1}\rho = (ya_1)(zb_1),\\
\sigma_2 &= \pi^{-1}\sigma_1\pi = (ya_2)(b_1b_2),\\
\sigma_3 &= \pi^{-1}\sigma_2\pi = (ya_3)(b_2b_3).\\
\sigma_4 &= \rho^m\sigma_1\rho^{-m} = (b_1b_2)(za_{k+1}), \qquad \text{where} \quad k = n - m \leqslant n - 3,\\
\sigma_5 &= \sigma_4\sigma_2 = (ya_2)(za_{k+1}),\\
\sigma_6 &= \pi^{-1}\sigma_5\pi = (ya_3)(b_1a_{k+2}),\\
\sigma_7 &= \rho^{m-1}\sigma_1\rho^{-(m-1)} = (b_2b_3)(za_{k+2}),\\
\sigma_8 &= \sigma_3\sigma_7 = (ya_3)(za_{k+2}),\\
\sigma_9 &= \sigma_6\sigma_8 = (za_{k+2}b_1), \text{ a 3-cycle.}
\end{aligned}$$

Now assume $n > m \geqslant 0$, $n \neq 2m$. If $m = 0, \pi^{-1}\rho = (a_1zy)$, a 3-cycle. If $m > 0$, consider

$$\begin{aligned}
\tau_1 &= \pi^{-1}\rho = (ya_1)(zb_1),\\
\tau_2 &= \rho^{-m}\tau_1\rho^m = (a_ma_{m+1})(zy),\\
\tau_3 &= \pi^{-2m}\tau_1\pi^{2m} = (yc)(a_ma_{m+1}),
\end{aligned}$$

where $c = a_1\pi^{2m} \notin \{y, z, a_m, a_{m+1}\}$. Then $\tau_2\tau_3 = (yzc)$ is a 3-cycle in $\langle\pi, \rho\rangle$.

The two cases considered above show that $\text{alt}(X) \subseteq \langle\pi, \rho\rangle$ unless: $m = n = 0$; $m = n = 1$; $n = 2$, $m = 1$; $n = m = 2$; or $n = 4$, $m = 2$. But trivially, $\text{alt}(X) \subseteq \langle\pi, \rho\rangle$ when $m = n = 0$ and $m = n = 1$. For completeness, we show that in the remaining cases $\text{alt}(X) \not\subseteq \langle\pi, \rho\rangle$.

Case (i): $n = 2, m = 1$. Without loss of generality, we may assume $X = \text{GF}(5)$ (the field with five elements) and let $y = 0, a_1 = 4, a_2 = 2, z = 1, b_1 = 3$. Consider

$$\Gamma_1 = \{x \mapsto ax + b : a, b \in \text{GF}(5), a \neq 0\},$$

a permutation group of order 20. With this identification, the permutation $x \mapsto 3x$ of Γ_1 coincides with π, and $x \mapsto 2x + 4$ coincides with ρ. Hence $\langle\pi, \rho\rangle \subseteq \Gamma_1$, and it is easy to see that $\langle\pi, \rho\rangle = \Gamma_1$.

Case (ii): $n = m = 2$. We may take $X = \text{GF}(5) \cup \{\infty\}$ and let $y = \infty, a_1 = 0, a_2 = 1, z = 2, b_1 = 3, b_2 = 4$. Consider

$$\text{PSL}_2(5) = \left\{x \mapsto \frac{ax + b}{cx + d} : a, b, c, d \in \text{GF}(5), ad - bc = \pm 1\right\},$$

a permutation group on X of order 60. Here the permutation $x \mapsto x + 1$ coincides with π, and $x \mapsto 1/(x + 1)$ coincides with ρ.

Thus

$$\langle \pi, \rho \rangle \subseteq \mathrm{PSL}_2(5),$$

and it is not difficult to see that in fact equality holds.

Case (iii): $n = 4, m = 2$. We take X to be the vector space of ordered triples of elements of GF(2) and let $y = 000, a_1 = 001, a_2 = 010, a_3 = 100, a_4 = 011, z = 110, b_1 = 111, b_2 = 101$. Consider the group Γ_2 of all affine transformations $x \mapsto xM + c$ where $c \in X$ and M is a non-singular 3×3 matrix over GF(2), a permutation group of order 1344. Here the permutations

$$x \mapsto x \begin{pmatrix} 011 \\ 100 \\ 010 \end{pmatrix} \quad \text{and} \quad x \mapsto x \begin{pmatrix} 010 \\ 101 \\ 011 \end{pmatrix} + (001)$$

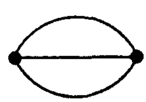
coincide, respectively, with π and ρ. Hence $\langle \pi, \rho \rangle \subseteq \Gamma_2$, and it can be shown that equality holds.

4. Proof of Theorem 2

The following lemma was introduced by H. Whitney [5], and a proof can also be found in [4, p. 85]. (G. N. Robertson has suggested that the statement be called the "Handle Theorem.") Here an *arc* is a finite tree with exactly two monovalent vertices (its *ends*).

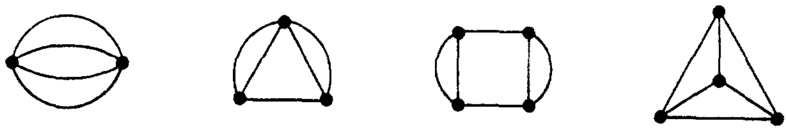
LEMMA 5. *Let G be a non-separable graph and K a non-separable proper subgraph of G with non-empty edge set. Then we can write $G = H \cup A$, where H is a non-separable subgraph of G containing K, A is an arc-subgraph of G, and $H \cap A$ consists only of the ends of A.* (Clearly, $\beta(G) = \beta(H) + 1$.)

The polygons are subdivisions of the (non-simple) graph consisting of a single loop and its incident vertex. Adding a "handle" to a polygon, we obtain the θ-graphs (non-separable graphs with $\beta(G) = 2$), which are subdivisions of the (non-simple) graph:



Adding a handle to a θ-graph, we see that every non-separable graph G

with $\beta(G) = 3$ is a subdivision of one of the four graphs:



The trivalent vertices of a θ-graph are its *nodes*. By the θ-graph of *type* $(j, k, l), j \geqslant k \geqslant l \geqslant 0$, we mean the θ-graph in which the three arc-subgraphs joining its nodes have, respectively, j, k, and l internal vertices. Such a θ-graph is simple if and only if $k \geqslant 1$.

We remark at this point that every simple non-separable graph G with $\beta(G) = 3$ contains a θ-graph whose type is *not* (2, 2, 1). This can be seen by considering the possible ways to add a handle to a θ-graph of type (2, 2, 1). We omit the details.

As we have observed in Section 2, it will suffice to show that

$$\operatorname{alt}(V(G) - \{x\}) \subseteq \Gamma(x)$$

for some particular vertex x of a finite simple non-separable graph G, not a polygon or θ_0.

We first prove Theorem 2 for θ-graphs. Let G be a finite simple θ-graph, let y and z be the nodes of G, and A_1, A_2, A_3 the three arcs of G joining y and z. Suppose A_2 has an internal vertex x, and let

$$p_1 = (z, a_n, \ldots, a_2, a_1, y),$$

$$p_2 = (y, b_m, \ldots, b_2, b_1, z),$$

$$q = (x, c_1, c_2, \ldots, c_s, y),$$

$$r = (x, d_1, d_2, \ldots, d_t, z),$$

be simple paths contained in A_1, A_3, A_2, A_2, respectively. Then

$$\pi = \sigma_{qp_2p_1\bar{q}} = (a_1 \cdots a_n z b_1 \cdots b_m),$$

and

$$\rho = \sigma_{rp_1p_2\bar{r}} = (b_1 \cdots b_m y a_1 \cdots a_n)$$

belong to $\Gamma(x)$.

Assume for the moment that the arcs A_1, A_2, A_3 have been indexed so that A_2 has an internal vertex, $n \geqslant 1$, but m and n are *not* 4, 2 or 2, 2 or 2, 1 in some order. This can be done unless G has type (4, 2, 2), (2, 2, 2), or (2, 2, 1) (these cases will be discussed separately). Then, by Lemma 4, any three vertices of $A_1 \cup A_3$ are the support of a 3-cycle of $\Gamma(x)$. By conjugating $(a_1 yz) \in \Gamma(x)$ by an appropriate power of

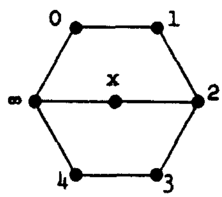
$$\tau = \sigma_{rp_2\bar{q}} = (d_1 \cdots d_t z b_1 \cdots b_m y c_s \cdots c_1),$$

we can find a 3-cycle in $\Gamma(x)$ containing a_1 and any given vertex of A_2, other than x. It is clear, then, that the 3-cycles in $\Gamma(x)$ generate a group which is transitive on $V(G) - \{x\}$. Hence, by Lemma 2, $\Gamma(x)$ contains $\operatorname{alt}(V(G) - \{x\})$.

Suppose G is the θ-graph of type (2, 2, 2) and take $m = n = 2$, $s = 1$, $t = 0$ in the definition of p_1, p_2, q, r above. Then $\pi^2\rho\pi^2\tau\pi^{-1}\rho\tau^{-1} = (zyc_1) \in \Gamma(x)$. Conjugating (zyc_1) by appropriate powers of π, and using Lemma 2, we see that $\operatorname{alt}(V(G) - \{x\}) \subseteq \Gamma(x)$.

Suppose G is the θ-graph of type (4, 2, 2), and take $n = 4$, $m = 2$, $s = 1$, $t = 0$ in the definition of p_1, p_2, q, r. Then $\rho\pi^{-1}\tau^{-2}\pi^{-1}\rho\tau^2 = (za_4a_1) \in \Gamma(x)$. By conjugating (za_4a_1) by appropriate powers of ρ and τ, $\operatorname{alt}(V(G) - \{x\}) \subseteq \Gamma(x)$ follows from Lemma 2.

Suppose G is θ_0, the θ-graph of type (2, 2, 1), and identify the vertices of G with $\{x, \infty\} \cup GF(5)$ as indicated below:



Here we claim that $\Gamma(x)$ coincides with

$$\mathrm{PGL}_2(5) = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathrm{GF}(5), ad - bc \neq 0 \right\},$$

a 3-transitive permutation group of order 120 acting on $\mathrm{GF}(5) \cup \{\infty\}$. Let $p = (x, \infty, 4, 3, 2, 1, 0, \infty, x)$, $q = (x, 2, 1, 0, \infty, x)$. Then p and q generate the fundamental group of G, so that $\sigma_p = (01234)$ and $\sigma_q = (\infty 012)$ generate $\Gamma(x)$. But the permutations $z \mapsto z + 1$ and $z \mapsto 3/(z + 3)$ of $\mathrm{PGL}_2(5)$ coincide, respectively, with σ_p and σ_q. Hence $\Gamma(x) \subseteq \mathrm{PGL}_2(5)$ and it can be checked that equality holds.

We now proceed by induction on $\beta(G)$. Let G be a finite simple nonseparable graph with $\beta(G) \geqslant 3$. Write $G = H \cup A$ where H is a nonseparable subgraph of G with $\beta(H) = \beta(G) - 1$ and A is an arc with only its ends in H. By a previous remark, H may be chosen *not* to be θ_0 (when $\beta(G) = 3$). With this understanding, $\operatorname{alt}(V(H) - \{x\}) \subseteq \Gamma_H(x)$ for each

$x \in V(H)$ by our discussion of θ-graphs above or by the induction hypothesis.

Let x and y be the ends of A. Since $\Gamma_G(x)$ contains $\Gamma_H(x)$ which contains 3-cycles, it will suffice by Lemma 3 to show that $\Gamma_G(x)$ is doubly transitive on $V(G) - \{x\}$.

Let $p = (x, a_t, a_{t-1}, \ldots, a_2, a_1, y)$ be the unique simple path from x to y in A. Let z be any vertex of H other than x, y. By the non-separability of H, there exists a simple path $q = (y, b_s, \ldots, b_2, b_1, x)$ from y to x in H which does not pass through z. Then $\sigma_{pq} = (ya_1a_2 \cdots a_tb_1b_2 \cdots b_s) \in \Gamma_G(x)$ and the i-th power of σ_{pq} takes y to a_i $(1 \leqslant i \leqslant t)$. Now $\Gamma_H(x)$ contains permutations fixing z and taking y to any vertex of H other than x and z. Hence the stabilizer of z in $\Gamma_G(x)$ is transitive on $V(G) - \{x, z\}$. Since $\Gamma_G(x)$ is also transitive on $V(G) - \{x\}$($\Gamma_H(x)$ contains a permutation taking y to z), we conclude that $\Gamma_G(x)$ is doubly transitive on $V(G) - \{x\}$.

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