Now recall that $\lambda = e^{2\pi i/3}$. So we have $\lambda^3 = 1$ and $\lambda^4 = \lambda$. Also, since $\lambda = \cos(2\pi/3) + i\sin(2\pi/3)$ and $\lambda^2 = \cos(4\pi/3) + i\sin(4\pi/3)$, a little bit of trigonometry shows that $\lambda^2 + \lambda = -1$.

Using these facts, we see that the coefficient of z^3 in the above equation is given by

$$2\lambda^2(\lambda^2 + \lambda) + 2\lambda^2 = 2\lambda^2(-1) + 2\lambda^2 = 0.$$

The coefficient of z^2 is

$$\lambda^4 + \lambda(\lambda^2 + \lambda) = \lambda - \lambda = 0.$$

And the final two terms are $\lambda^3 z - z = 0$. Therefore our equation $G_\lambda^3(z) - z = 0$ reduces to a polynomial of the form $Az^8 + Bz^7 + Bz^6 + Dz^5 + Ez^4 = 0$ and so we see that 0 is a root of order four for this equation. We know that our fixed point at 0 is one of those roots. The other fixed point for G_λ occurs at $z = 1 - e^{2\pi i/3}$, so this is a second root. The other six roots must then be points lying on cycles of period 3. But this equation shows that one of those cycles has merged with the fixed point at 0 when $\lambda = e^{2\pi i/3}$. Thus we see a similar merger of a period-3 cycle and a fixed point at the bifurcation value.

There is a video called "A Tour of the Mandelbrot Set" that displays the corresponding Julia sets as a parameter wanders around a path in and out of $\mathcal{M}$ which is available at `math.bu.edu/DYSYS/animations.html`.

17.4 Experiment: Periods of the Bulbs

As we have seen, each of the bulbs attached to the main cardioid C_1 in $\mathcal{M}$ meets the boundary of the main cardioid at a parameter value for which there is a neutral fixed point at which the derivative of Q_c is $\exp(2\pi ip/q)$. So this is a period-q bulb. Recall that we call this the p/q-bulb and denote it by $B_{p/q}$. Interestingly, you can determine the value of q by simply looking at the antenna attached to this bulb. The antenna attached to $B_{p/q}$ has a junction point to which there are exactly q spokes attached (this includes the spoke that connects directly to the bulb). Several such bulbs are displayed in Figure 17.9 as well as in Plate 18.

Goal: As we have seen, these bulbs are arranged around the boundary of the main cardioid in the exact order of the rational numbers between 0 and 1. Your goal in this experiment is to find a way to identify the entire fraction p/q in three different ways.

Procedure: First, just by looking at the spokes attached to the junction point in $B_{p/q}$, how do you determine the value of the numerator p? Second, use a computer to plot the filled Julia set corresponding to a parameter that lies close to the center of the bulb $B_{p/q}$. By looking at this set, can you

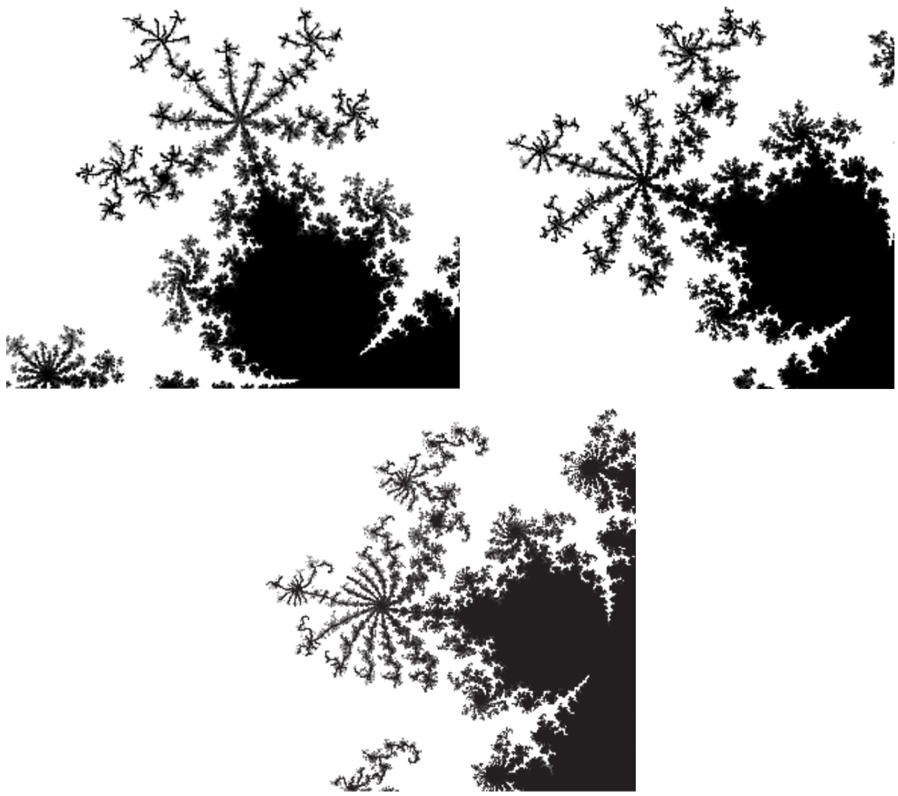

FIGURE 17.9
The period-8, -9, and -13 bulbs attached to the main cardioid.

again determine the fraction p/q? Third, again use the computer to plot in succession the orbit of the attracting cycle of period q that corresponds to the previous parameter. Watch how this orbit evolves as you iterate. How can you use this to determine p/q? In an essay, describe these three different methods for determining the fraction p/q associated with $B_{p/q}$.

Notes and Questions:

1. Using a computer, investigate the sub-bulbs hanging off the period-3 bulb $B_{1/3}$ attached to the main cardioid. Is there any relation between the ordering of the periods of these sub-bulbs and the periods of the $B_{p/q}$? Also, by looking at the antennas of these sub-bulbs, can you find a way to determine both the period of these sub-bulbs as well as the fact that they are attached to $B_{1/3}$? Some of these sub-bulbs are displayed in Plates 21 and 22. Explain what you find in a brief essay with some pictures.

2. Now repeat the previous investigation by looking at the period-4 bulb $B_{1/4}$.

3. Several videos showing the filled Julia sets that arise as a parameter winds around just outside the main cardioid can be found at math.bu.edu/DYSYS/animations.html.

17.5 Experiment: Periods of the Other Bulbs

Goal: In this experiment you will hunt for all of the bulbs in the Mandelbrot set that correspond to attracting cycles of period n (not just those attached to the main cardioid). As you perform this experiment, you will undoubtedly view more of the wonderfully intricate geometric shapes that abound in $\mathcal{M}$.

Procedure: Use a computer to select a c-value from a bulb and then determine the period of the corresponding attracting cycle. Find all bulbs that feature an attracting n-cycle for $n = 4$, 5, and 6 (and note that sometimes these bulbs will be the main cardioids of small copies of $\mathcal{M}$).

Results: Indicate the locations of the period-4, -5, and -6 bulbs on a copy of the Mandelbrot set. Some of these locations are indicated in Figure 17.10.

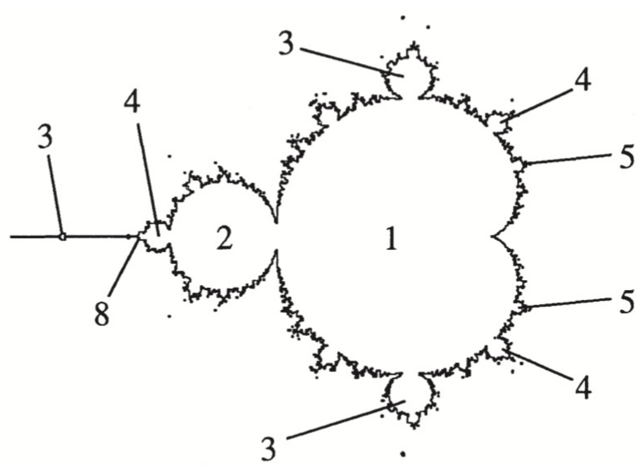

FIGURE 17.10
Periods of some of the bulbs in $\mathcal{M}$.

Notes and Questions:

1. There are exactly six period-4 bulbs.

2. Next find all of the period-5 bulbs. There are 15 of them. It is very difficult to find all of them, but it is fun to try!

3. Now try period 6, if you are a glutton for punishment. There are 27 of them! Good luck! While it may seem that these bulbs are ordered in no apparent fashion around $\mathcal{M}$, there is in fact a beautiful description of how all of these bulbs are arranged due to Douady and Hubbard [16]. Unfortunately, complete details of their work are too advanced for this text. See [28] for more details.

4. One fact that will considerably shorten the time it takes to perform this and some subsequent experiments is that the Mandelbrot set is symmetric about the real axis. Moreover, the periods of symmetrically located bulbs are the same. We ask you to verify this in exercise 2 at the end of this chapter. Thus, for this experiment, you need only check the decorations in the upper half-plane.

17.6 Experiment: How to Add

Goal: The aim of this experiment is to show how the concept of "Farey addition" from the area of mathematics known as number theory arises when viewing the arrangement of the principal bulbs $B_{p/q}$ attached to the main cardioid of $\mathcal{M}$. Geometrically, the goal is to determine inductively the largest bulb between two given bulbs attached to C_1.

Procedure: To start, let's identify the cusp of the main cardioid as the place where the 0/1-bulb originates. Then we have, on the far-left side of the main cardioid on the negative real axis, the place where 1/2-bulb is attached. Which is the largest bulb between the 1/2-bulb and the 0/1-bulb in the upper half-plane? Clearly, it is the 1/3-bulb. Now continue this process. Let a_1 be the fraction associated with the largest bulb between the 1/2-bulb and the 1/3-bulb. Then let a_2 be the fraction associated with the largest bulb between the two previous bulbs, namely, the 1/3 and a_1-bulbs. Then let a_3 be the fraction associated with the largest bulb between the previous two, the a_1- and a_2-bulbs. And continue...

Notes and Questions:

1. Do you recognize the sequence generated by the denominators of $0/1, 1/2, 1/3, a_1, a_2, a_3, \ldots$?

2. Can you find a method to determine the fraction corresponding to the largest bulb between the a_k and a_{k+1}-bulbs in this sequence? This method is called "Farey addition."

3. A brief video exploring zooms into the bulbs following this "Farey sequence" is available at `math.bu.edu/DYSYS/animations.html`.

4. Now for some number theory. What is the fraction a_1 between 1/2 and 1/3 with the smallest denominator? And what is the fraction a_2 between a_1 and 1/3 with the smallest denominator? And how about a_3 between a_1 and

a_2? Do you see a similar pattern here? This is where Farey addition arises in number theory.

Warning: Students are absolutely forbidden to use Farey addition in the real world. Only folks with a Ph.D. in mathematics are allowed to add fractions this way. Things become so much easier once you get your Ph.D. in math!

17.7 Experiment: Find the Julia Set

Goal: The aim of this experiment is to show you how each decoration in the Mandelbrot set corresponds to c-values whose Julia sets are qualitatively similar.

Procedure: Figures 17.11 and 17.12 depict eight different filled Julia sets. Your task is to identify the bulb in the Mandelbrot set that contains the

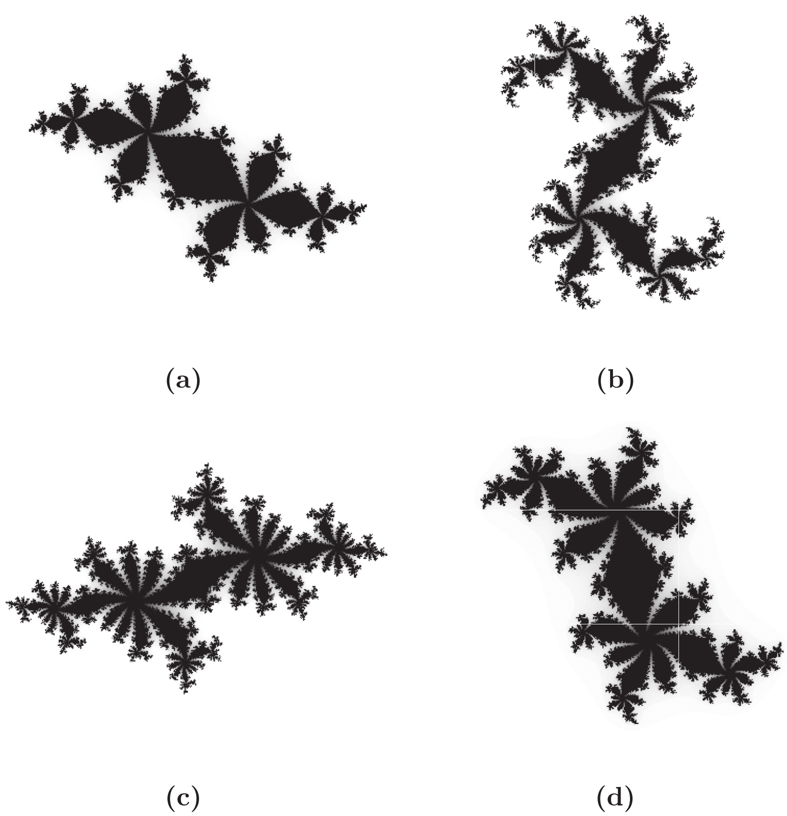

FIGURE 17.11
Find these Julia sets in the Mandelbrot set.

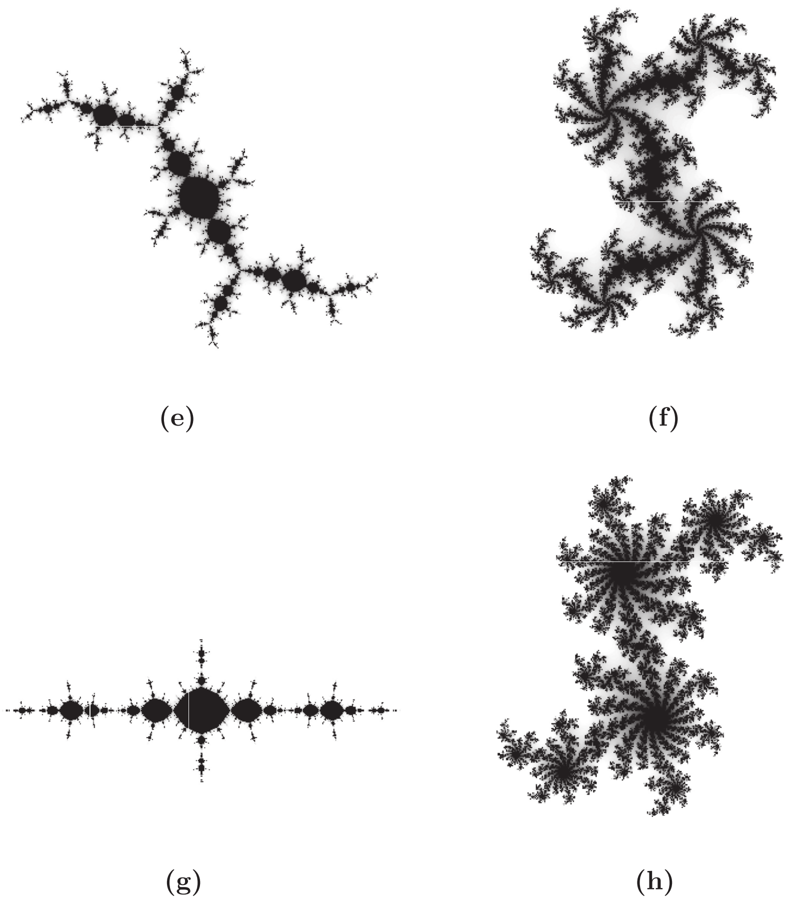

FIGURE 17.12
More Julia sets to find in the Mandelbrot set.

c-value that produced these Julia sets. In each case, use the computer to select c-values from a bulb in the Mandelbrot set and then draw the corresponding Julia set.

Results: On an image of the Mandelbrot set, indicate the approximate locations of Julia sets "a" through "h" in Figures 17.11 and 17.12.

Notes and Questions:

1. As a hint, each of the Julia sets (with only two exceptions) comes from bulbs directly attached to the main cardioid. You may have to hunt within a given decoration for the exact Julia set depicted, but you will quickly see that

all filled Julia sets from a single decoration have essentially the same features, that is, are qualitatively the same.

2. Do you recognize any connection between the structure of the Julia sets from a given decoration and the period of that decoration, as described in the previous experiments?

3. Once you have found the locations of each of these Julia sets, investigate the shapes of the Julia sets in corresponding locations in smaller copies of the Mandelbrot set, particularly the period-3 Mandelbrot set on the real axis. Is there any relation between these images? Explain in a brief essay.

17.8 Experiment: Similarity of the Mandelbrot Set and Julia Sets

Goal: The goal of this experiment is to view a remarkable relationship between the structure of the Mandelbrot set near special c-values and the shape of the corresponding filled Julia sets.

Procedure: For a variety of different bulbs $B_{p/q}$ attached to the main cardioid, magnify the junction point of the antenna from which all of the spokes emanate. Zoom in on this junction point several times. Then choose the corresponding c-value. Finally, use the computer to draw the filled Julia set for this c-value. An example of this is displayed in Plates 19 and 20.

Results: Compare the structure of the magnified Mandelbrot set near the junction point with a magnified portion of the corresponding filled Julia set. Are there any similarities between these two images? Explain in an essay.

Notes and Questions: The similarity between magnified portions of the Mandelbrot set and the corresponding filled Julia sets holds only near certain c-values such as the central junctions of the antenna. These are c-values for which 0 is eventually periodic. Such c-values are called *Misiurewicz points*.

Exercises

1. Prove that Q_c has an attracting 2-cycle when c lies inside the circle of radius 1/4 centered at -1.

2. Prove that the Mandelbrot set is symmetric about the real axis. *Hint:* Show this by proving that Q_c is conjugate to $Q_{\overline{c}}$. Show that your conjugacy takes 0 to 0. Therefore the orbit of 0 has similar fates for both Q_c and $Q_{\overline{c}}$.

The Logistic Functions. The following six exercises deal with the logistic family $F_\lambda(z) = \lambda z(1 - z)$, where both λ and z are complex numbers.