Organization of the Mandelbrot Set

These notes are an almagamation of notes from Devaney's book, and includes information from the video below, which has fantastic animations.

`https://www.youtube.com/watch?v=oNxPSP2tQEk`
`patreon.com/mathstown`

The Main Cardiod

The main cardiod of the Mandelbrot set consists of the values of c so that the orbit of 0 becomes eventually fixed. To find the equation of the cardiod, there are two equations to solve- One for fixed points, and one to be "atracting".

$$z^2 + c = z \qquad 2|z| \leq 1$$

Put these together, and the second equation says that our z values lie in a disk of radius 1/2. Solve the first equation for c: $c = z - z^2$, and now put them together by taking $z = \frac{1}{2}\mathrm{e}^{2\pi i\theta}$. Therefore, the equation of the main cardiod (parameterized by θ) is given by

$$\frac{1}{2}\mathrm{e}^{2\pi i\theta} - \frac{1}{4}\mathrm{e}^{4\pi i\theta}$$

In conclusion, inside the main cardiod, the orbit of zero will be attracted to a period 1 point.

The next region to the left

The next region to the left will be the set of c that have an attracting period 2 point (for the orbit of 0). The equations that define this region are:

$$(z^2 + c)^2 + c = z \qquad 2(z^2 + c)(2z) = 4z(z^2 + c) \Rightarrow \quad 4z(z^2 + c) = D$$

As before, the equation on the right defines a unit disk D. The first equation looks complicated, but remember that we can divide out the polynomial used for the fixed points:

$$(z^2 + c)^2 + c = z \quad \Rightarrow \quad (z^2 - z + c)(z^2 + z + (c + 1)) = 0 \quad \Rightarrow \quad z^2 + z + (c - 1) = 0$$

Solve this equation with the restriction that $4a(z^2 + c) = D$[1]. We get that

$$c = -1 + \frac{1}{4}D$$

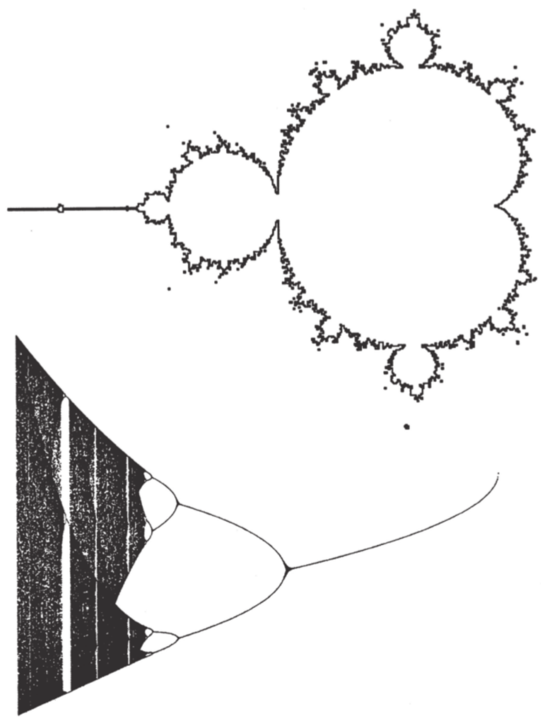
The object we see to the left of the main cardiod is therefore a circle of radius 1/4, centered at $c = -1$. If we intersect the circle with the real line, we get the interval $[-5/4, -3/4]$, which is the region of c for which the real function $Q_c(x) = x^2 + c$ had an attracting period two point (refer to the bifurcation diagram we had earlier).

The remaining "bulbs" on the Mandelbrot do not have perfect geometric shapes (in terms of circles or cardiods), so we'll only get the equations for these first two.

[1] This is a nonlinear system of two equations with two unknowns, z and c, so we use a computer algebra system to solve it

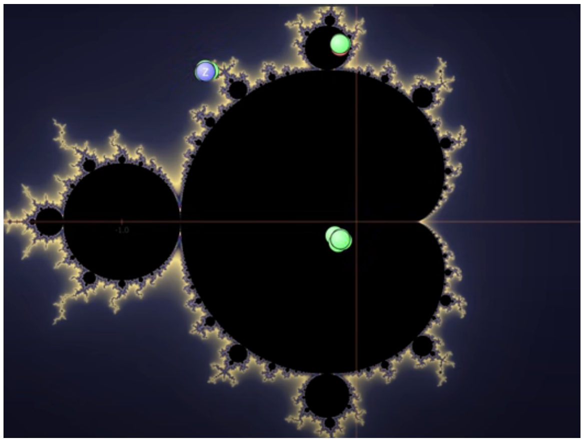
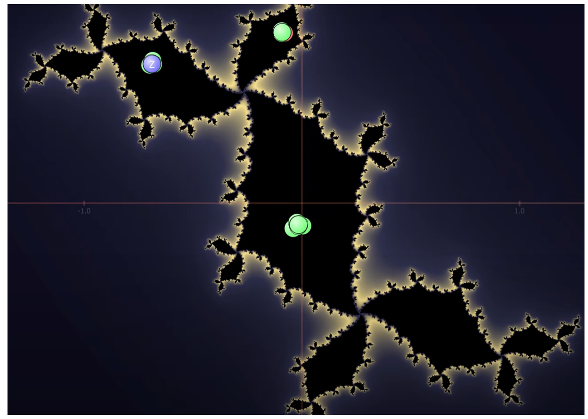
Other bulbs along the real axis

As we further move left along the real axis, the behavior of the orbit of 0 is the same as the bifurcation diagram for the function $Q_c(x) = x^2 + c$. That is, initially the periods undergo a period doubling bifurcation. See the figure below (you might also recall the appearance of the period three window in the bifurcation diagram and see what it corresponds to on the Mandelbrot set).



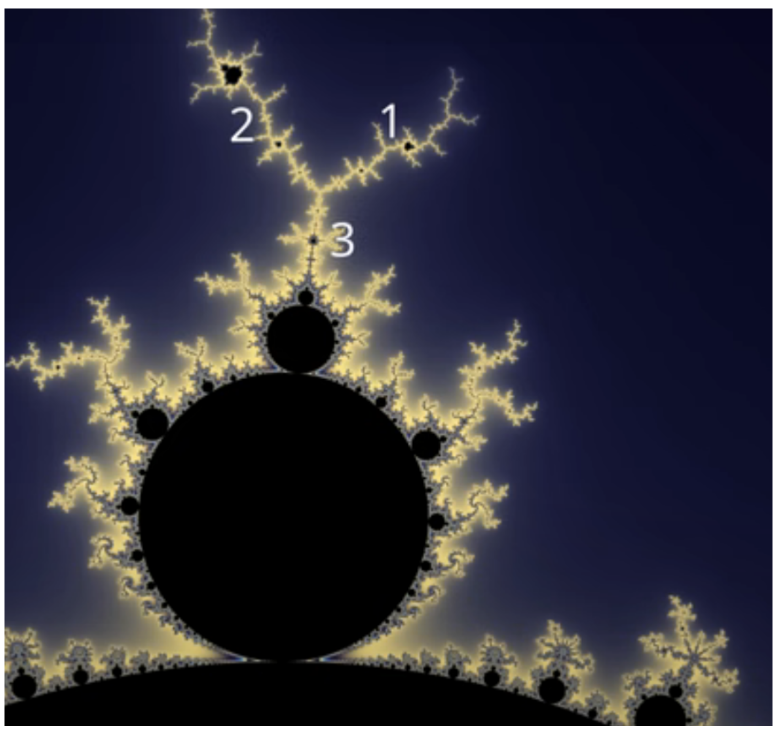
The Period 3 Bulb

The period 3 bulbs are towards the top and bottom of the main cardiod. These (filled) bulbs represent values of c that correspond to connected Julia sets where the orbit of 0 is attracted to a period three point. It is interesting to see the period three point plotted on the Mandelbrot set (the $c-$plane) and the orbit of 0 plotted on the Julia set- This is where you might want to check the video out. I'll include a couple of screenshots below.

Signposts on the bulbs of the main cardiod

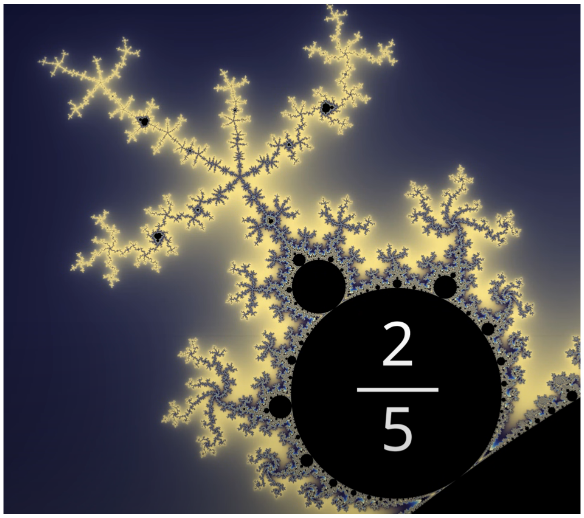
If we look closer at the period three bulb, we see three antennae coming from the main bulb (see how to count these- Don't start with the main stalk, go counterclockwise to the first antenna from there. See the screenshot below to see how to count these for the period three bulb.



Now we're going to associate each bulb to a rational number (lowest form): m/n. The main period three bulb shown above will be 1/3 (the value 1 coming from the location of the shortest antenna).

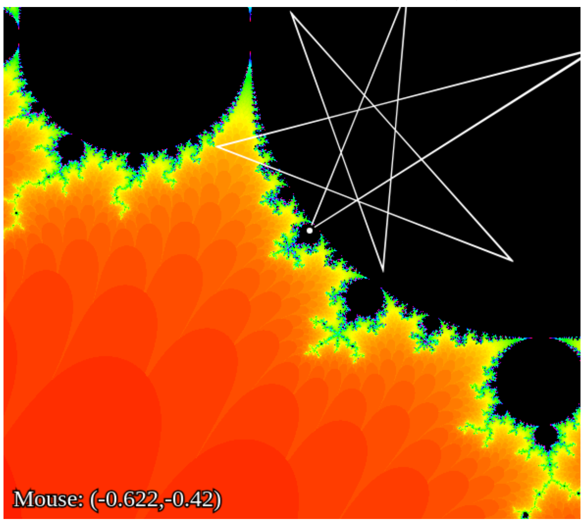
Period 5 bulb

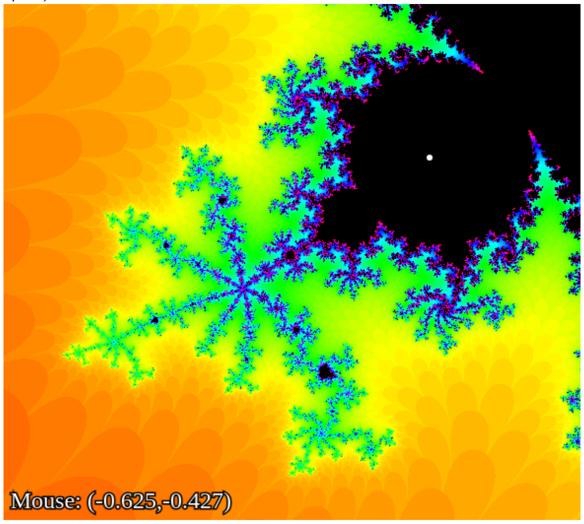
Notice that the period 5 bulb indeed has five antenna (counting the main stalk), and now the shortest one is in position 2. This will correspond to the 2/5 bulb.



Period 7 bulb

One more to see if we can read the signpost on the bulb. Below[2] we see the orbit of 0 using a c value from a period 7 bulb (the orbit is given in white). Next to it is a zoomed shot. What would m, n be for this bulb? (Answer: 4/7)

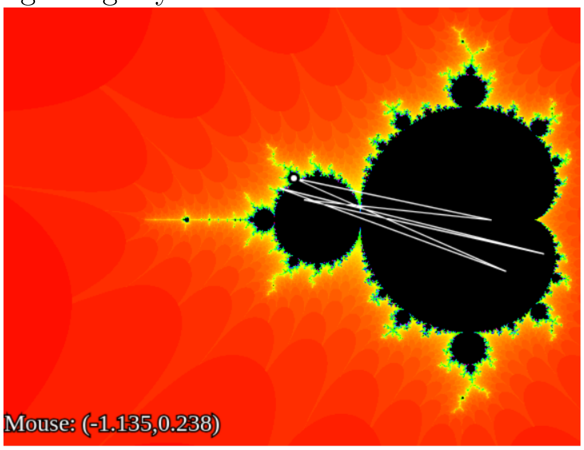


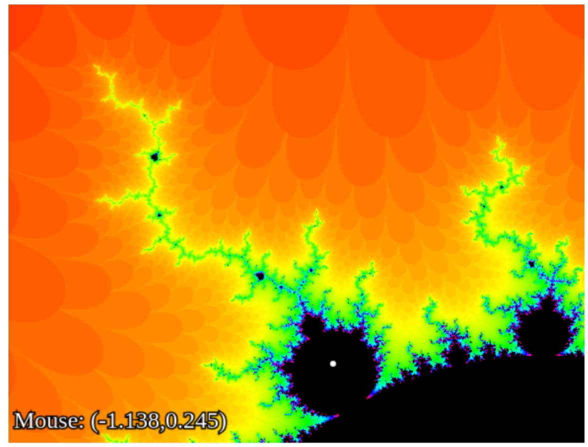


[2]These screenshots are using applets embedded in the free online book available through LibreTexts called "Complex Analysis - A Visual and Interactive Introduction (Ponce Campuzano)", Section 5.5, By Juan Carlos Ponce Campuzano, University of Queensland.

Off the main cardiod

Below is a large bulb off of the **period two** bulb rather than the main cardiod. The first image shows that the attracting orbit is a period six point, but on the left, using a zoom, we see only three antenna. As you may have guessed, the rule for the period of the bulb changes slightly in this bulb- We have to multiply by two to get the correct period.





Branching off the main cardiod bulbs

If we zoom into the large period 3 bulb at the top of the cardiod, and continue moving c up through the attached bulbs, we'll see the following period progression:

$$3, \quad 2 \cdot 3, \quad 2^2 \cdot 3, \quad 2^3 \cdot 3, \quad \ldots$$

Similarly, zooming into the $2/5$ bulb whose primary period was 5, and moving straight through the attached bulbs, they have a similar progression:

$$5, \quad 2 \cdot 5, \quad 2^2 \cdot 5, \quad 2^3 \cdot 5, \quad \ldots$$

It seems like this kind of progression persists with every bulb attached to the main cardiod.

Off of the main cardiod onto the period 2 bulb, if we move along to the next largest bulb (period 6, as described previously), then we can probably predict the progession of periods:

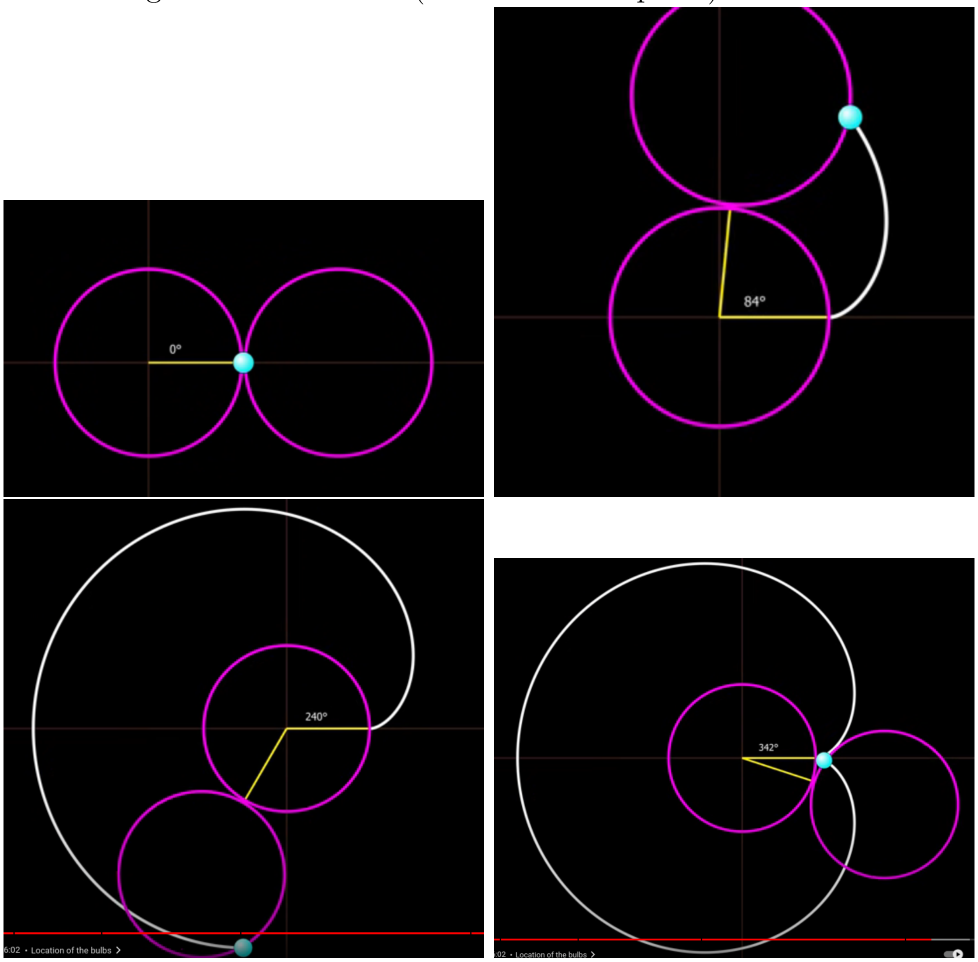
$$2, \quad 6, \quad 12, \quad 24, \ldots$$

or

$$2 \cdot 1, \quad 2 \cdot 3, \quad 2^2 \cdot 3, \quad 2^3 \cdot 3, \quad \ldots$$

Finishing up the main cardiod

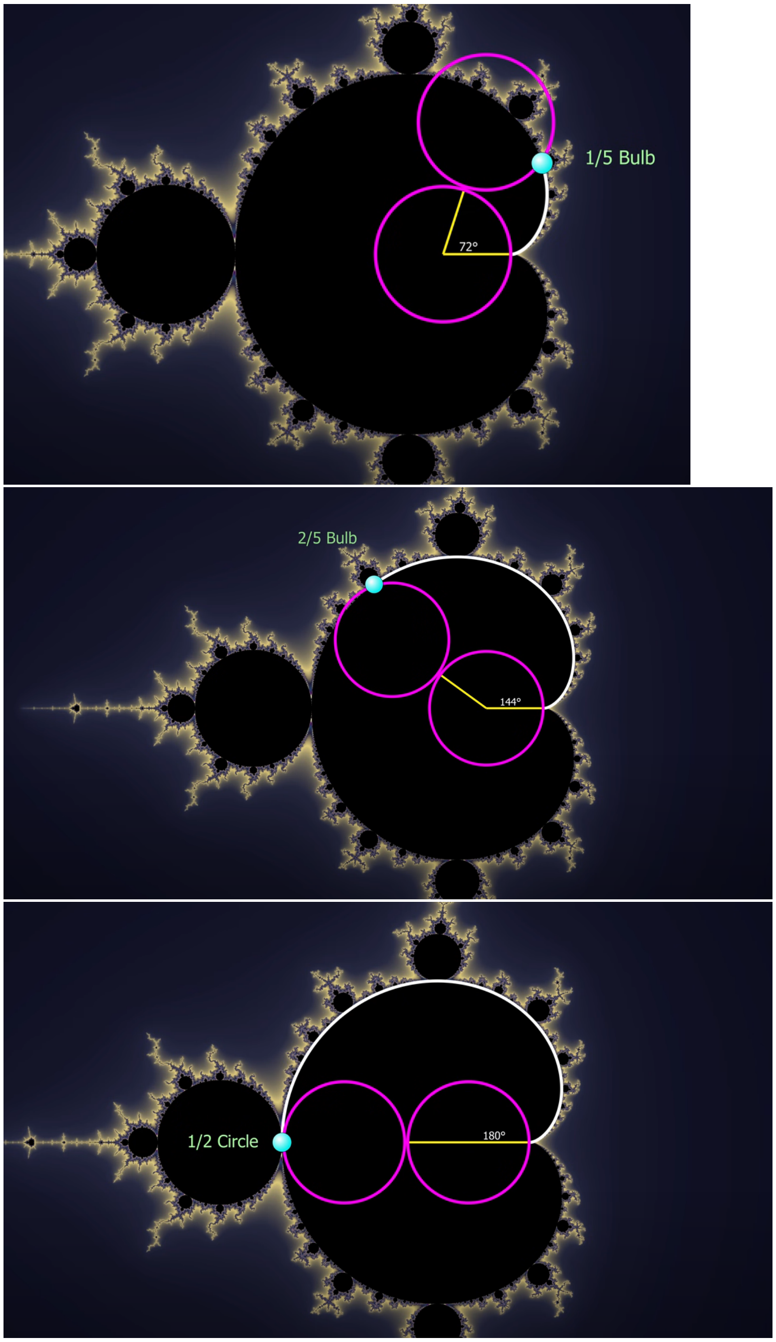
We've discussed how to determine the period of the main bulb off of the cardiod, but we haven't discussed how the bulbs are ordered. A cardiod can be constructed by fixing one circle, then rotating another circle around it. Here is the progression below, starting at angle 0, then moving around the circle (follow the blue point):

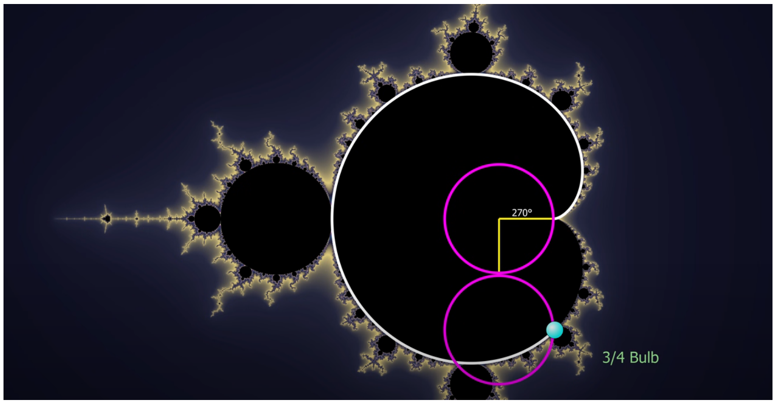


The angle that the blue point makes, as shown, is called the **internal angle**, and as expected, it starts at 0 and ends at 360. Think about each of these angles as sweeping past some rational fraction of the whole circle.

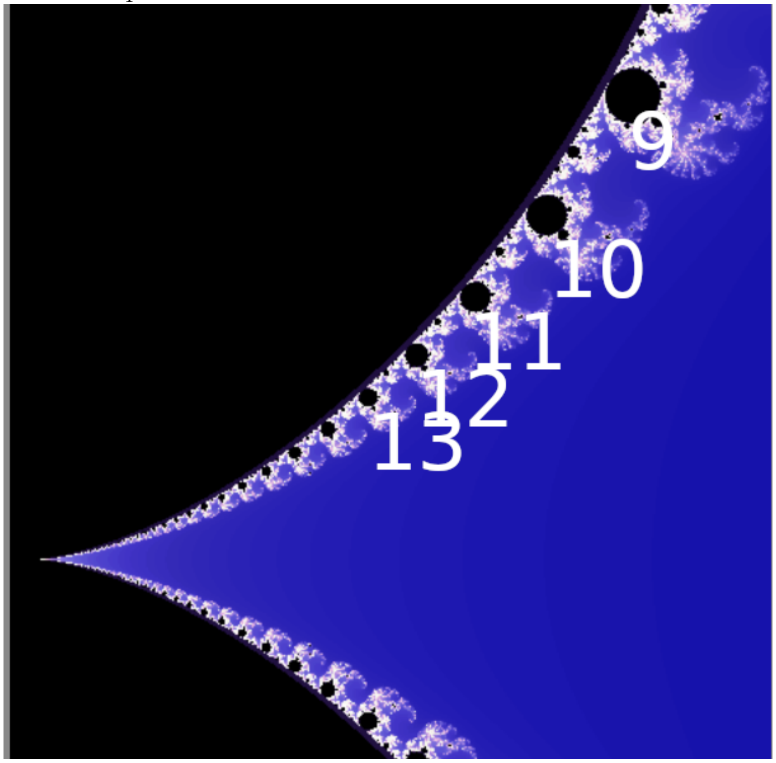
For example, 45 degrees is $\frac{1}{8}$ of a circle, 90 degrees is $\frac{1}{4}$ of a circle, and so on. Similarly, $\frac{2}{5}$ of a circle is at 144 degrees, and so on.

Suprisingly, if we're at the point on the cardiod corresponding to $\dfrac{m}{n}$ of a circle (in lowest form), then the period of the bulb is n. Several examples are shown below.


1/5 Bulb
72°
2/5 Bulb
144°
1/2 Circle
180°



As you might predict, those bulbs that were attached to the main cardiod close to the point $c = -\frac{1}{4} + 0i$ are increasing in period as shown below.



Concluding Remarks

There are a lot more interesting features of the Mandelbrot set- We've only touched the surface. If you have found this interesting, I would encourage you to do more exploration around the Mandelbrot set.

Homework: The Labs in Ch 17

You should try Lab 17.7 in particular- It will give you some experience looking at the detail of Julia sets along the main cardiod.

Here's a good "Mandelbrot Explorer" for the lab in order to match Julia sets:

`https://mandel.gart.nz/`

This applet is also very good- you can zoom into the Mandelbrot set, and if you hover the mouse, the applet will tell you what the period is.

`https://mandelbrot.page/`