Solutions to Review Questions, Exam 3

1. A child is flying a kite. If the kite is 90 feet above the child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the cord forms a line- actually an unrealistic assumption).

   Note: As with all word problems, your notation may be different than mine First, draw a picture of a right triangle, with height 90, hypotenuse $y(t)$, the other leg $x(t)$. Note that these are varying until after we differentiate. So, we have that
   $$(y(t))^2 = (x(t))^2 + 90^2$$
   and
   $$2y(t)\frac{dy}{dt} = 2x(t)\frac{dx}{dt}$$
   We want to find $\frac{dy}{dt}$ when $y = 150$ and $\frac{dx}{dt} = 5$. We need to know $x(t)$, so use the first equation:
   $$150^2 - 90^2 = x^2 \Rightarrow x = 120$$
   so
   $$2 \cdot 150\frac{dy}{dt} = 2 \cdot 120 \cdot 5 \Rightarrow \frac{dy}{dt} = 4$$

2. Use differentials to approximate the increase in area of a soap bubble, when its radius increases from 3 inches to 3.025 inches ($A = 4\pi r^2$)
   $$dA = 8\pi r\, dr$$
   and $r = 3$, $dr = 0.025$, so
   $$dA = 8 \cdot 3 \cdot 0.025 \cdot \pi = 0.6\pi$$
   We can check our estimate by looking at the exact increase in area: At 3 inches, the area is 36π, at 3.025 inches, the area is 36.6025π, so the exact change in area is 0.6025π

3. True or False, and give a short reason:

   (a) If air is being pumped into a spherical rubber balloon at a constant rate, then the radius will increase, but at a slower and slower rate.
   True. $V = \frac{4}{3}\pi r^3$, so
   $$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$
   Since $\frac{dV}{dt}$ is constant, lets call it k. Solving for $\frac{dr}{dt}$, we get:
   $$\frac{dr}{dt} = \frac{k}{4\pi r^2}$$
   so as r gets large, $\frac{dr}{dt}$ gets smaller (but does stay positive).

   (b) If $y = x^5$, then $dy \geq 0$
   Before answering, look at dy:
   $$dy = x^4\, dx$$
   So, as long as dx is positive, $dy \geq 0$. However, dx (or Δx) can be negative, which would make dy negative. So, the answer is False.

   (c) If a car *averages* 60 miles per hour over an interval of time, then at some instant, the speedometer must have read exactly 60.
   True. This is exactly the Mean Value Theorem. The velocity (call it $f'(c)$) at some point in the time interval must have been equal to the average velocity between times a and b,
   $$\frac{f(b) - f(a)}{b - a}$$
   (we can assume the position function is differentiable).

(d) A global maximum is always a local maximum. False, since endpoints cannot be local maximums (by definition, we ruled those out).

(e) The linear function $f(x) = ax + b$, where a, b are constant, and $a \neq 0$, has no minimum value on any open interval. (An interval is open if it does not include its endpoints).

True. If $a \neq 0$, then $f(x)$ is not a horizontal line, which means it must go up or down in the interval- so *if* the endpoints were included, then one of the endpoints would be the min (or max). But since the endpoints are NOT included, the function will never actually "hit" its minimum.

(f) Suppose P and Q are two points on the surface of the sea, with Q lying generally to the east of P. It is possible to sail from P to Q (always sailing roughly east), without *ever* sailing in the exact direction from P to Q.

This is meant to get you to think about the Mean Value Theorem, and it is false. While sailing from P to Q, at some point the boat MUST have been pointed in the direction from P to Q.

(g) If $f(x) = 0$ has three distinct real solutions, then $f'(x) = 0$ must have (at least) two solutions.

If f is differentiable, then this is true. If f has three real solutions, let's call them a, b, c, then the Mean Value Theorem says that:

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0 \text{ for some } c \in [a, b]$$

and

$$\frac{f(c) - f(b)}{c - b} = f'(u) = 0 \text{ for some } u \in [b, c]$$

4. Linearize at $x = 0$:

$$y = \sqrt{x}e^{x^2}(x^2 + 1)^{10}$$

To linearize means to replace y by the equation of the tangent line, so we need a point and a slope. The point for the tangent line is where $x = 0$, $y = 0$.

Now we need the slope at $x = 0$. To take the derivative, let's use logarithmic differentiation:

$$\ln(y) = \ln(x^{1/2}) + \ln(e^{x^2}) + \ln((x^2 + 1)^{10}) = \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2 + 1)$$

Differentiate:

$$\frac{1}{y}y' = \frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}$$

And the last step is to multiply both sides by y, then substitute in for y:

$$y' = \sqrt{x}e^{x^2}(x^2 + 1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}\right)$$

so we get $y' = 0$. The line is therefore $L(x) = 0$ (A lot of work for zero!)

5. Estimate by linear approximation the change in the indicated quantity.

(a) The volume, $V = s^3$ of a cube, if its side length s is increased from 5 inches to 5.1 inches.

$$dV = 3s^2\,ds \Rightarrow dV = 3 \cdot 25 \cdot 0.1 = 7.5$$

(b) The volume, $V = \frac{1000}{p}$, of a gas, if the pressure p is decreased from 100 to 99.

$$dV = \frac{-1000}{p^2}\,dp \Rightarrow dV = \frac{-1000}{100^2}(-1) = 0.1$$

(c) The period of oscillation, $T = 2\pi\sqrt{\frac{L}{32}}$, of a pendulum, if its length L is increased from 2 to 2.2.

First, note that:

$$T = \frac{2\pi}{\sqrt{32}}\sqrt{L}$$

which makes it slightly easier to differentiate:

$$dT = \frac{\pi}{\sqrt{32}}L^{-1/2}\,dL \Rightarrow dT = \frac{\pi}{\sqrt{64}} \cdot 0.2 \approx 0.0785$$

6. For the following problems, find where f is increasing or decreasing. If asked, also check concavity.

(a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ (Also check for concave up/down)

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

Build a chart:

x	$-$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$	$+$
	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$

Now, f is increasing on $-1 < x < 0$ and $x > 2$, and f is decreasing for $x < -1$ and $0 < x < 2$.
Now, for concavity:

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

This is not factorable into "nice" numbers, but we can use the quadratic formula to see that:

$$3x^2 - 2x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{7}}{3}$$

so that $f''(x)$ might change sign at those points:

$3x^2 - 2x - 2$	$+$	$-$	$+$
	$x < \frac{1-\sqrt{7}}{3}$	$\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$	$x > \frac{1+\sqrt{7}}{3}$

so f is concave down for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$, concave up everywhere else (except at the roots, where $f''(x) = 0$).

(b) $f(x) = \frac{x}{x+1}$

$$f'(x) = \frac{1 \cdot (x+1) - 1 \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

so f is always increasing ($f'(x)$ is always positive).
For concavity, we check the second derivative:

$$f''(x) = -2(x+1)^{-3} = \frac{-2}{(x+1)^3}$$

We could build a chart, or, by inspection, we see that if $x < -1$, the denominator is negative, so overall f'' is positive: For $x < -1$, f is concave up.
For $x > -1$, the denominator is positive, so overall f'' is negative: For $x > -1$, f is concave down.

(c) $f(x) = x\sqrt{x^2+1}$

$$f'(x) = \sqrt{x^2+1} + x(\frac{1}{2}(x^2+1)^{-1/2}2x) = \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}$$

From this, we see that every term in the expression is positive, so f is always increasing. To check the second derivative, it might be useful to simplify first:

$$f'(x) = \sqrt{x^2+1}\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}$$

Now computing the second derivative:

$$f''(x) = \frac{4x \cdot \sqrt{x^2+1} - (4x^2+1)\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{4x(x^2+1) - x(4x^2+1)}{\sqrt{x^2+1}}}{x^2+1} = \frac{3x}{(x^2+1)^{3/2}}$$

The denominator is always positive, but the numerator changes sign at the origin, so f is concave down if $x < 0$, concave up if $x > 0$.

7. Show that the given function satisfies the hypotheses of the Mean Value Theorem. Find all numbers c in that interval that satisfy the conclusion of that theorem. For comparison purposes, given these functions and intervals, what would the Intermediate Value Theorem conclude? Finally, find the global max and global min for each function.

   (a) $f(x) = x^3$, $[-1, 1]$

   Since f is a polynomial, it is continuous and differentiable everywhere. In particular, it will satisfy the MVT on $[-1, 1]$.

   First, we see that $f'(c)$ will be $3c^2$, so we need to solve:

$$3c^2 = \frac{f(1) - f(-1)}{1 - (-1)} = 1$$

   so $c = \sqrt{\frac{1}{3}} \approx 0.5773$.

   (b) $f(x) = \sqrt{x - 1}$, $[2, 5]$

   The function f will be continuous on its domain, $x > 1$. The derivative is $\frac{1}{2}(x-1)^{-1/2}$, which exists for $x > 1$. Therefore, f is continuous on $[2, 5]$ and differentiable on $(2, 5)$. Now we apply the MVT:

$$\frac{1}{2}(c-1)^{-1/2} = \frac{2-1}{5-2} = \frac{1}{3}$$

   so that

$$\frac{1}{\sqrt{c-1}} = \frac{2}{3} \Rightarrow \sqrt{c-1} = \frac{3}{2} \Rightarrow c - 1 = \frac{9}{4} \Rightarrow c = \frac{13}{4} = 3.25$$

   (c) $f(x) = x + \frac{1}{x}$, $[1, 5]$

   First, f is continuous for $x > 0$, and the derivative:

$$f'(x) = 1 - x^{-2}$$

   also exists for $x \neq 0$. So the MVT will apply on the interval $[1, 5]$. We compute $f(5) = 5 - \frac{1}{5} = \frac{24}{5}$ and $f(1) = 1 + 1 = 2$, and so:

$$1 - c^{-2} = \frac{\frac{24}{5} - 2}{5 - 1} = \frac{7}{10}$$

   so that

$$c^{-2} = \frac{3}{10} \Rightarrow c^2 = \frac{10}{3} \Rightarrow c = \pm\sqrt{\frac{10}{3}} \approx \pm 1.825$$

   We choose the value of c in $[1, 5]$, so $c = \sqrt{\frac{10}{3}}$

8. Show that $f(x) = x^{2/3}$ does not satisfy the hypotheses of the mean value theorem on $[-1, 27]$, but nevertheless, there is a c for which:

$$f'(c) = \frac{f(27) - f(-1)}{27 - (-1)}$$

   Find the value of c.

   First, $f(x) = x^{2/3}$ is a continuous function, but its derivative is $f'(x) = \frac{2}{3}x^{-1/3}$, which does not exist at $x = 0$.

   Let's see if we can find a suitable c anyway:

$$\frac{2}{3}c^{-1/3} = \frac{9 - (-1)}{27 - (-1)} = \frac{10}{28}$$

$$c^{-1/3} = \frac{15}{28} \Rightarrow c^{1/3} = \frac{28}{15} \Rightarrow c = \left(\frac{28}{15}\right)^3 \approx 6.504$$

9. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

   (a) The trucker claimed that he hadn't been speeding. Is this possible? Explain. Nope. Not possible. The trucker went 100 miles in 1.25 hours, which is not possible if you go (at a maximum) of 65 miles per hour (which would only get you (at a max) 81.25 miles).

   (b) The fine for speeding is $35.00 plus $2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine? By the last computation, the trucker had an *average* speed of 80 mph, so we can guarantee (by the MVT) that at some point, the speedometer read exactly 80. So, this gives $35.00 + $2.00 (15) = $65.00

10. Let $f(x) = \frac{1}{x}$

   (a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$? Since f is continuous on this closed interval, there is a global max and global min (on the interval).

   (b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT? The EVT was stated on an interval of the form $[a, b]$, which implies that we cannot allow a, b to be infinite.

11. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \le f'(x) \le 1$ for all x. Use the Mean Value Theorem to explain why $f(2)$ cannot be 3.

   We can form the "average velocity":

   $$\frac{f(2) - f(0)}{2 - 0} = \frac{f(2)}{2}$$

   Now, if $f(2)$ were equal to 3, then the MVT would state that $f'(c) = \frac{3}{2}$ for some c between 0 and 2. But, $f'(x)$ must be less than 1 for all x. EXTRA: Can you compute the largest and smallest values for $f(2)$?

   From the previous computation and the restrictions on $f'(x)$, we see that $f(2)$ might be any number between 1 and 2, but no larger than 2, and no less than 1.

12. Sketch the graph of a function that satisfies all of the given properties:

   $$f'(-1) = 0, f'(1) \text{ does not exist}, f'(x) < 0 \text{ if } \|x\| < 1, f'(x) > 0 \text{ if } \|x\| > 1$$

   $$f(-1) = 4, f(1) = 0, f''(x) > 0 \text{ if } x > 0$$

   We'll show this in class.

13. Use Newton's Method to compute x_1, x_2, x_3 if $f(x) = x^3 - x^2 - 1$, and $x_0 = 1$.

   [This won't be on the exam]

14. Problem 92, p. 270

   The water skier is moving up the ramp (she's traveling along the hypotenuse of a right triangle). Draw a picture of the right triangle, and let $z(t)$ be how far along the hypotenuse she is. Let $y(t)$ be her height at z. Then, you should have similar triangles, where the full base of the triangle is 15, the full height is 4, the partial distance along the hypotenuse is z.

   By the Pythagorean Theorem, the full hypotenuse is $\sqrt{4^2 + 15^2} = \sqrt{241}$, and by similar triangles, we have:

   $$\frac{y}{z} = \frac{4}{\sqrt{241}} \Rightarrow y = \frac{4}{\sqrt{241}} z$$

   Now we are given that $\frac{dz}{dt} = 30$, and we want to find how fast she is rising when $z = 4$, or what is $\frac{dy}{dt}$ when $z = 4$. Differentiating, we get:

   $$\frac{dy}{dt} = \frac{4}{\sqrt{241}} \frac{dz}{dt} = \frac{120}{\sqrt{241}} \approx 7.71 \text{ feet per second}$$

15. Section 4.9 will not be on the exam.

16. Local maxs and mins of $f(x) = x + \sqrt{1-x}$ using the first and second derivative tests.

    To use the first derivative, we know that if f is differentiable, and f' changes sign from positive to negative, then f is increasing then decreasing, so we have a local maximum. Similarly, if f' changes sign from negative to positive, we have a local minimum. Use a sign chart to determine these.

    For the second derivative, if $f'(a) = 0$ and $f''(a) > 0$, then the function is concave up at a. This means that $x = a$ is where a local minimum occurs. Similarly, if $f'(a) = 0$ and $f''(a) < 0$, then the function is concave down at $x = a$, and we have a local maximum at $x = a$. Notice that to use the second derivative, we don't need a sign chart.

    First we differentiate:
    $$f'(x) = 1 - \frac{1}{2\sqrt{1-x}}, \quad f''(x) = -\frac{1}{4}(1-x)^{-3/2}$$

    Now, the critical points are $x = \frac{3}{4}$ and $x = 1$. Using a sign chart, we see that $f'(x) > 0$ if $x < \frac{3}{4}$, and $f'(x) < 0$ if $\frac{3}{4} < x < 1$. From the first derivative changing sign from positive to negative, we know that at $x = \frac{3}{4}$ we have a local maximum.

    From the second derivative, $f''\left(\frac{3}{4}\right) = -2$ which means that f is concave down at $3/4$, so again we see that there is a local maximum there.

17. Find a cubic function $y = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

    [This won't be on the exam- You'll have 4 equations in a, b, c, d, and we could solve by substitution, but it could take a while]

18. A metal storage tank is to hold a certain volume, V. It is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal? (Note: There is a bottom to the tank, and you should consider V to be some given constant- so your answer might depend on V).

    [This won't be on the exam]

19. If you want more practice with some related rates problems, see the handout on our Calculus web page.