Quick Sheet: Continuity

1. Definition: A function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

2. This definition says three things: (i) $f(a)$ can be computed. (ii) $\lim_{x \to a} f(x)$ exists. (iii) The values in (i) and (ii) are the same.

   - To show that f is continuous at $x = a$, show that the three properties above hold.
   - To show that f is NOT continuous at $x = a$, show which of the three parts do not hold.

3. Theorem: All of our basic functions are continuous *on their domains*. That is, all polynomials, rational functions, sine, cosine, tangent (and their inverses), exponential functions, logarithms, root functions are all continuous on their domain.

   > Are ALL functions continuous on their domains? No, but probably all of the functions you are familiar with are. A template function for something not continuous anywhere: The function that takes the value 0 on the rational numbers, 1 on the irrationals. (Property (i) holds, but property (ii) does not).

   Furthermore, all (finite) sums, differences, products, quotients, and compositions of continuous functions are continuous *on their new domains*.

   We use this theorem when we "find where f is continuous". That is, if f was formed by one of the previous operations on our basic functions, we can restate the problem as "find the domain of f".

4. Definition (Continuity on a closed interval): We say that f is continuous on a closed interval $[a, b]$ if f is continuous on (a, b), and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

   This is a "global" definition of continuity. The original definition was a "local" definition of continuity.

5. The Intermediate Value Theorem:

   - The formal statement: If f is continuous on $[a, b]$ and N is a value between $f(a)$ and $f(b)$, then there is a c in $[a, b]$ such that $f(c) = N$.
   - The intuitive meaning: The image of a closed interval under a continuous function is a closed interval (that is, if the domain is a closed interval, the range must be a solid interval with no breaks).
   - The main useage: If $f(a) > 0$ and $f(b) < 0$, then there must be a solution to $f(x) = 0$ for some x in (a, b). (The same holds if $f(a) < 0$ and $f(b) > 0$).