

Take Home Quiz Solutions

1. A boat leaves a dock at 2PM traveling due south at 20 kilometers per hour. Another boat is traveling due east at 15 kilometers per hour, and arrives at the dock at 3PM.

(a) At 2PM, how far apart are the boats? At 2PM, the boat traveling south is at the dock. The boat traveling east takes an hour to get to the dock traveling at 15km/hr. Therefore, this boat is 15km away from the dock.

The boats are 15km apart.

(b) At 3PM, how far apart are the boats? At this point, the boat traveling east has arrived at the dock, but the southbound boat is 20km away.

The boats are 20km apart.

(c) At 2:30PM, how far apart are the boats? The boat going south has traveled 10km south. The eastward bound boat is 7.5km closer to the dock than before, so it is $15-7.5=7.5$ km away.

The distance apart is (by the Pythagorean Theorem):

$$\sqrt{10^2 + 7.5^2} = 12.5\text{km}$$

(d) At t minutes past 2PM, how far away is the eastbound boat from the dock? How far away is the southbound boat?

The purpose of the previous 3 problems was to get you to think about how to solve this problem.

At t minutes past 2PM, the southbound boat will have traveled:

$$20 \frac{\text{km}}{\text{hr}} \cdot t \text{ hours} = 20 \frac{\text{km}}{\text{hr}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot t \text{ min} = \frac{1}{3}t \text{ km}$$

So, if t is measured in minutes, then at time t , the southbound boat has traveled $\frac{1}{3}t$ kilometers. We can check this: after 60 minutes, the boat has traveled $\frac{1}{3}60 = 20\text{km}$

For the eastbound boat, we have a similar computation. At t minutes, the eastbound boat will travel this many kilometers:

$$15 \frac{\text{km}}{\text{hr}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot t \text{ min} = \frac{1}{4}t \text{ km}$$

BUT, it started 15 kilometers away, so its distance from the dock is:

$$15 - \frac{1}{4}t \text{ km}$$

(e) At how many minutes past 2PM were the boats closest to each other? Let $D(t)$ be the squared distance between the boats. We will minimize this quantity for $0 \leq t \leq 60$ minutes:

$$D(t) = \left(\frac{1}{3}t\right)^2 + \left(15 - \frac{1}{4}t\right)^2 = \frac{25}{144}t^2 - \frac{15}{2}t + 225$$

Now, differentiate and set the derivative to zero:

$$D'(t) = \frac{25}{72}t - \frac{15}{2} = 0 \Rightarrow t = \frac{108}{5} = 21.6$$

At this value of time, how far apart are the boats?

$$D(21.6) = \frac{25}{144}(21.6)^2 - \frac{15}{2} \cdot 21.6 + 225 = 144$$

so the boats are 12 km apart. Comparing this value to $t = 0$, which we computed as 15, and at $t = 60$, which we computed as 20, we see that this is the minimum distance apart.

An Alternative: Minimize the Actual Distance:

This is messy, but we'll show that we do get the same answer. In this case,

$$D(t) = \sqrt{\frac{25}{144}t^2 - \frac{15}{2}t + 225}$$

Now,

$$D'(t) = \frac{1}{2} \left(\frac{25}{144}t^2 - \frac{15}{2}t + 225 \right)^{-1/2} \cdot \left(\frac{25}{72}t - \frac{15}{2} \right) = \frac{\frac{25}{72}t - \frac{15}{2}}{2 \cdot \sqrt{\frac{25}{144}t^2 - \frac{15}{2}t + 225}}$$

Now, set that to zero and solve:

$$\frac{\frac{25}{72}t - \frac{15}{2}}{2 \cdot \sqrt{\frac{25}{144}t^2 - \frac{15}{2}t + 225}} = 0 \Rightarrow \frac{25}{72}t - \frac{15}{2} = 0$$

by multiplying both sides by $2 \cdot \sqrt{\frac{25}{144}t^2 - \frac{15}{2}t + 225}$. Note that this is the same equation as before.

2. A cylinder is constructed so that its radius is equal to its height. In this case, its volume can be computed as $V = \pi r^3$. Calculate ΔV and dV , if the radius changes from 2.0 to 2.1.

Recall that in general, if $y = f(x)$, and x changes from x to $x + \Delta x$, then we define:

$$\Delta y = f(x + \Delta x) - f(x) \text{ The Actual Change of } y$$

and

$$dy = f'(x)dx \text{ The Approximate Change of } y$$

Now in this case,

$$\Delta V = V(2.1) - V(2) = \pi (2.1^3 - 2^3) \approx 3.9615$$

On the other hand,

$$dV = 3\pi r^2 dr = 3\pi \cdot 2^2 \cdot 0.1 = 1.2\pi \approx 3.7699$$