Exam II Review Solutions

1. Short Answer:

   (a) State the Extreme Value Theorem: Let f be continuous on $[a, b]$. Then f attains a maximum and a minimum on the interval $[a, b]$.

   (b) What is the restriction on the domain of the sine function so that it is invertible? On the tangent function? On the cosine function?

   For the sine function, the standard restriction is $[-\pi/2, \pi/2]$. For the tangent function, $(-\pi/2, \pi/2)$. For the cosine, we take the standard restriction to be $[0.\pi]$

   (c) What is the procedure for finding the maximum or minimum of a function $y = f(x)$ on a closed interval, $[a, b]$.

   We find the critical points for f; that is, where $f'(x) = 0$ or where $f'(x)$ does not exist. We build a table with these values and the endpoints, and evaluate f at these points.

   (d) What is the procedure for finding a local maximum or minimum?

   We look at the critical points, and then check to see where f is increasing/decreasing. If f' changes from positive to negative, then that x value is where f has a local maximum. If f' changes from negative to positive, then there is a local minimum.

   (e) What is the procedure for finding a global maximum or minimum (assume the domain is all reals)?

   Check the critical points and evaluate the sign chart for f'. There are too many cases to do this for a general function- take it on a case-by-case basis (we did lots of examples in class).

2. True or False, and give a short reason:

   (a) If $f'(a) = 0$, then there is a local maximum or local minimum at $x = a$.

   FALSE. The sign of f' must change at a to have a local maximum or minimum.

   (b) $\sin^{-1}(x) = \dfrac{1}{\sin(x)}$

   FALSE. The notation $\sin^{-1}(x)$ is reserved to denote *the inverse function of the sine.* If we want the reciprocal, we will write either $\csc(x)$ or $(\sin(x))^{-1}$

   (c) $\tan(\tan^{-1}(x)) = x$ for all x

   TRUE. This is a property of composing a function with its inverse, $f(f^{-1}(x)) = x$, for all x in the domain of $f^{-1}(x)$.

   We could also say that $f^{-1}(f(x)) = x$, for all x in the (restricted) domain of f.

   (d) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x) + g(x)$ is increasing.

   TRUE. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$. If these are both positive, so is the sum.

(e) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x)g(x)$ is increasing.
FALSE. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$. It is possible that $g(x)$ and $f(x)$ are negative, in which case the derivative of the product would also be negative.

(f) If $f(x)$ is increasing, and $g(x)$ is decreasing, then $f(g(x))$ is decreasing.
TRUE. Since $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$, and $f'(a) > 0$ for all a, and $g'(x) > 0$ for all x, the derivative of the composition is also positive.

3. Maximums and Minimums and related questions.

(a) Find two numbers whose difference is 100 and whose product is a minimum.
Let x, y be the two numbers. Then we want to find the minimum of $P = xy$ if $|x - y| = 100$. We could write this as two inequalities,

$$x - y = 100 \text{ or } y - x = 100$$

In either case,

$$P = x(100 - x) \text{ or } P = x(100 + x)$$

so:

$$P'(x) = 100 - 2x \text{ or } P'(x) = 100 + 2x$$

so that the critical points are either $x = 50$ or $x = -50$.
In the first case, $x = 50$, then $y = 100 - 50 = 50$, and the product is 2500. In the second case, $x = -50$ and $y = -50$, so the product again is 2500.

(b) A window in the shape of a rectangle for the base is surmounted by a half-circle (see the figure). If the perimeter must be 30 feet, find the dimensions of the window that gives the maximum amount of area (to maximize the amount of light).
The area consists of the upper half circle, which is $\frac{1}{2} \cdot \pi r^2$, and the area of the rectangle, which will be the length of base times the height, $2rh$. Therefore,

$$A = \frac{\pi}{2}r^2 + 2rh$$

with the perimeter:

$$30 = \frac{2\pi r}{2} + 2r + 2h = (\pi + 2)r + 2h$$

Now we can find h in terms of r for the substitution:

$$\frac{30 - (\pi + 2)}{2}r = h \quad \Rightarrow \quad h = 15 - \frac{\pi + 2}{2}r$$

Now perform the substitution so that Area is in terms of r only:

$$A(r) = \frac{\pi}{2}r^2 + 2r \cdot \left(15 - \frac{\pi + 2}{2}r\right) = 30r - \frac{\pi + 4}{2}r^2$$

Now physical constraints on r: If $r = 0$, all the material is used for the "rectangle", and if $(\pi + 2)r = 30$, all the material is used for the semicircle. In the first case, the area is 0, and in the second case, $h = 0$, $r = \frac{30}{\pi+2}$, and the area is:

$$\frac{1}{2}\pi\left(\frac{30}{\pi+2}\right)^2 \approx 53.477$$

We have evaluated the area at the endpoints, now we get the critical points:

$$A'(r) = 30 - (\pi + 4)r = 0 \quad r = \frac{30}{\pi + 4}$$

and evaluating the area here gives:

$$A\left(\frac{30}{\pi+4}\right) = \frac{450}{\pi+4} \approx 63.01$$

Therefore, the dimensions of the window that give the maximum light are:

$$r = \frac{30}{\pi+4} \approx 4.2 \text{ feet}, \quad h = 15 - \frac{\pi+2}{2}r = \frac{30}{\pi+4}$$

(c) Find a positive number such that the sum of the number and its reciprocal is as small as possible.

Let x be the number. Then we want to find the minimum of:

$$F(x) = x + \frac{1}{x}, \quad x > 0$$

We differentiate to find the critical points:

$$F'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$$

The point -1 is outside of our interval, so we consider only 1. Do a sign chart to find that:

$$F'(x) < 0 \text{ if } 0 < x < 1, \quad F'(x) > 0 \text{ if } x > 1$$

Our conclusion is that F is decreasing for $0 < x < 1$, and increasing for $x > 1$. Thus there is a global minimum for $x + 1/x$ at $x = 1$.

(d) Find the dimension of the rectangle of largest area that can be drawn if the base of the rectangle is on the $x-$axis and its other vertices are on the parabola $y = 8 - x^2$ (See the figure).

We want to find the maximum of $2xy$, where $y = 8 - x^2$. (Did you remember to take $2x$ as the length of the full base of the rectangle?).

Therefore, we make our substitution and determine an interval for x: We want to find the maximum of:

$$F(x) = 2x(8 - x^2) = 16x - 2x^3, \quad 0 \le x \le \sqrt{8}$$

(We get $\sqrt{8}$ by looking at the $x-$intercepts of the parabola). We note that $F(0) = F(\sqrt{8}) = 0$, and now we find the critical point:

$$F'(x) = 16 - 6x^2 = 0 \Rightarrow x = \pm\sqrt{\frac{8}{3}}$$

We disregard the negative value. Now, the area of this rectangle is:

$$16 \cdot \sqrt{\frac{8}{3}} - 2\left(\sqrt{\frac{8}{3}}\right)^3 \approx 17.418$$

The dimensions of the rectangle that give this maximum area are: The base has length $2\sqrt{8/3}$, and the height is $16/3$.

(e) Find the dimensions of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 and 4, as shown in the Figure.

There are a couple of approaches to this problem. One way to do it is to put the coordinate system at the right angle. This way, the hypotenuse goes through the points $(0, 3)$ and $(4, 0)$, and the equation of the hypotenuse is:

$$y = -\frac{3}{4}x + 3$$

Now, we want to find the maximum of the area of the rectange, xy, with y given by the equation of the line:

$$A(x) = x\left(-\frac{3}{4}x + 3\right), \quad 0 \le x \le 4$$

The area at $x = 0$ and $x = 4$ is zero, and those are the endpoints. We now look for the critical points:

$$A'(x) = -\frac{3}{2}x + 3 = 0 \Rightarrow x = 2$$

The area in this case is:

$$A(2) = 2(3/2) = 3$$

and the base has length 2, height has length $3/2$.

(f) Find the point on the (sideways) parabola $y^2 = 2x$ closest to the point $(1, 4)$. Hint: It suffices to find the minimum of the distance squared.

The distance formula squared:

$$D = d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

We have a point on the parabola, (x, y) and the point $(1, 4)$,

$$D = (x - 1)^2 + (y - 4)^2$$

Now, $2x = y^2$, so $x = \frac{1}{2}y^2$:

$$D(y) = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

We have no constraints on y, so we check the critical points and see where D is increasing/decreasing:

$$D'(y) = 2\left(\frac{1}{2}y^2 - 1\right) \cdot y + 2(y-4) = y^3 - 8$$

The only critical point is $y = 2$. By considering the sign chart, if $y < 2$, $D'(y) = y^3 - 8 < 0$ and if $y > 2$, $D'(y) > 0$. Therefore, we have a global minimum at $y = 2$, and $x = 2$.

(g) Find the global maximum and minimum of the given function on the interval provided:

i. $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$

Check the value of f at the endpoints and the critical points.

$$f'(x) = \frac{1}{2}(9 - x^2)(-2x) = \frac{-x}{\sqrt{9-x^2}}$$

We have critical points at $x = \pm 3$ and $x = 0$. Of these, only $x = 0$ is inside the interval. Now, evaluate f at $x = -1, x = 2, x = 0$:

$$f(-1) = \sqrt{8} = 2\sqrt{2} \approx 2.82, \quad f(2) = \sqrt{5} \approx 2.23, \quad f(0) = 3$$

The maximum of 3 occurs at $x = 0$, and the minimum of 2.23 occurs when $x = 2$.

ii. $g(x) = x - 2\cos(x)$, $[-\pi, \pi]$

Find the critical points:

$$g'(x) = 1 + 2\sin(x) = 0 \Rightarrow \sin(x) = \frac{-1}{2}$$

From the $30 - 60 - 90$ triangle, we see that $x = \frac{\pi}{6}$ will give $\sin(\pi/6) = \frac{1}{2}$. From the unit circle, we get that $x = -\frac{\pi}{6}$ or $\frac{7\pi}{6}$. However, we want to express this as an angle between $-\pi$ and π, so we re-write $7\pi/6$ as $-5\pi/6$.

We now evaluate g at $x = -\pi, \pi, -\pi/6, -5\pi/6$. Doing this, we get the approximations (in order):

$$-1.14, 5.14, -2.25, -0.88$$

Therefore, the minimum of -2.25 occurs at $x = -\pi/6$, and the maximum of 5.14 occurs at $x = \pi$.

iii. $h(x) = x^2 + \frac{2}{x}$, $\left[\frac{1}{2}, 2\right]$
The critical points of h are;

$$h'(x) = 2x - \frac{2}{x^2} = 0 \Rightarrow x = 1$$

We could say that 0 is also a critical point, but it is not in the domain of h, and it is outside the interval of interest. We evaluate h at $x = 1/2, 1, 2$ which gives, in order, $17/4, 3, 5$. Therefore, the minimum of 3 occurs when $x = 1$, and the maximum of 5 occurs when $x = 5$

(h) Find the regions where f is increasing/decreasing:

i. $f(x) = x^3 - 12x + 1 \Rightarrow f'(x) = 3x^2 - 12$
The critical points are $x = \pm 2$, and using a sign chart,

$(3x^2 - 12)$	$+$	$-$	$+$
	$x < -2$	$-2 < x < 2$	$x > 2$

Therefore, f is increasing for $x < -2$ and $x > 2$, and f is decreasing if $-2 < x < 2$.

ii. $g(x) = x - 2\sin(x)$ for $0 < x < 3\pi$. Here, $g'(x) = 1 - 2\cos(x)$ so the critical points are found where $\cos(x) = \frac{1}{2}$. Using the $30 - 60 - 90$ triangle, we see that this is where $x = \pi/3$. From the unit circle, we get $x = \pi/3$ and $-\pi/3$. We want to rewrite these so that they are inside the interval $[0, 3\pi]$, so we get that the critical points are:

$$\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

We do a sign chart:

$1 - 2\cos(x)$	$-$	$+$	$-$	$+$
	$0 < x < \pi/3$	$\pi/3 < x < 5\pi/3$	$5\pi/3 < x < 7\pi/3$	$7\pi/3 < x < 3\pi$

or it may be simpler just to graph $1 - 2\cos(x)$ to get these values. From this, we see that g is decreasing in the first and third intervals, and f is increasing in the 2d and 4th intervals.

iii. $h(x) = \frac{x}{(1+x)^2}$

$$h'(x) = \frac{1 - x}{(1 + x)^3}$$

so the critical point is $x = 1$. We'll also keep an eye on $x = -1$, even though it was not in the domain of h. A sign chart:

$(1 - x)$	$+$	$+$	$-$
$(1 + x)^3$	$-$	$+$	$+$
	$x < -1$	$-1 < x < 1$	$x > 1$

From this we see that h is increasing on $-1 < x < 1$, and h is decreasing otherwise.

(i) Set up the expressions to find the maximum area of the rectangle that can be inscribed in a circle of radius 3. Do not solve.

Let the upper right vertex of the rectangle be (x, y). Then the area of the rectangle will be:

$$A = 2x \cdot 2y = 4xy$$

From the formula for the circle,

$$x^2 + y^2 = 3^2$$

we can solve for y in terms of x:

$$y = \sqrt{9 - x^2}$$

We disregard the negative root since we're looking at the upper half of the circle. This gives us the setup of the problem: Find the maximum of the following function on the given interval:

$$A(x) = 4x\sqrt{9 - x^2}, \quad 0 \leq x \leq 3$$

4. Differentiate:

(a) $y = \sin(3x)$ Use the chain rule: $y' = \cos(3x) \cdot 3 = 3\cos(3x)$.
Note that in general, if $y = \sin(kx)$, then $y' = k\cos(kx)$

(b) $y = \arctan(x)$ Recall this one from our table of derivatives (that is, we want to have memorized this). $y' = \dfrac{1}{1 + x^2}$

(c) $y = \sqrt{3x} - \cos(x^2) \quad y' = \dfrac{1}{2}(3x)^{-1/2} \cdot 3 + \sin(x^2) \cdot 2x = \dfrac{3}{2\sqrt{3x}} + 2x\sin(x^2)$

(d) $y = \sin(2x)\cos(3x)$ Use the product rule:

$$y' = \cos(2x) \cdot 2 \cdot \cos(3x) + \sin(2x)(-\sin(3x)) \cdot 3 = 2\cos(2x)\cos(3x) - 3\sin(2x)\sin(3x)$$

(e) $f(x) = \arcsin(1 - 3x)$ Use the chain rule:

$$f'(x) = \frac{1}{\sqrt{1 - (1 - 3x)^2}} \cdot -3 = \frac{-3}{\sqrt{6x - 9x^2}}$$

(f) $f(x) = \tan(x^2 - 3x + 4)$ Use the chain rule:

$$f'(x) = \sec^2(x^2 - 3x + 4) \cdot (2x - 3) = (2x - 3)\sec^2(x^2 - 3x + 4)$$

(g) $y = \sec(x)$ Rewrite: $y = \frac{1}{\cos(x)} = (\cos(x))^{-1}$. Take the derivative using the chain rule:

$$y' = -(\cos(x))^{-2} \cdot -\sin(x) = \frac{\sin(x)}{\cos^2(x)}$$

This would be a good answer. Later, we'll rewrite this so that

$$y' = \sec(x)\tan(x)$$

(h) $y = \sin^2(3x-5)$

$$y' = 2\left(\sin(3x-5)\right)^1 \cdot \cos(3x-5) \cdot 3 = 6\sin(3x-5)\cos(3x-5)$$

(i) $y = \arctan(\arcsin(\sqrt{x}))$

$$y' = \frac{1}{1+(\arcsin(\sqrt{x}))^2} \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$

Not much to simplify here, except $x = (\sqrt{x})^2$.

5. Other Trig Questions: For the solutions, see the handwritten files on the class web site.

(a) Find the remaining trig ratios, if:

i. $\tan(\alpha) = 2$, α in 1st Quadrant

ii. $\sec(\phi) = -3/2$, ϕ is in the 2d Quadrant

iii. $\cot(\beta) = 3$, $\pi < \beta < 2\pi$

(b) Find all values in $[0, 2\pi]$ that satisfy the inequality:

i. $\sin(x) \leq 1/2$

ii. $-1 < \tan(x) < 1$

iii. $2\cos(x) + 1 > 0$

iv. $\sin(x) > \cos(x)$

(c) Find the exact value of each expression:

i. $\tan^{-1}(\sqrt{3})$

ii. $\arcsin(1)$

iii. $\sin\left(\sin^{-1}\left(\frac{7}{10}\right)\right)$

iv. $\arcsin(-1/\sqrt{2})$

v. $\tan(\tan^{-1}(325))$

(d) Simplify using a triangle:

i. $\sin(\cos^{-1}(x))$

ii. $\sin(\tan^{-1}(x))$

iii. $\tan(\sin^{-1}(x))$