Calculus I Review Solutions

1. Compare and contrast the three "Value Theorems" of the course. When you would typically use each.

   The three value theorems are the Intermediate, Mean and Extreme value theorems. The intermediate and extreme value theorems only require the function to be continuous on a closed interval. The mean value theorem additionally requires f to be differentiable on the open interval.

   (a) We use the Mean Value Theorem for problems relating the derivative of an interior point to the values of f at the endpoints. For example, if f has two roots at $x = a$, $x = b$, then we know f must have a critical point between a and b. A common use is to substitute $f'(c)(b - a)$ in place of $f(b) - f(a)$.

   (b) The Intermediate Value Theorem says that for every $y-$value w between $f(a)$ and $f(b)$, there is an x in $[a, b]$ so that $f(x) = w$. We can use this to prove the existence of solutions to $f(x) = 0$, if $f(a)$ and $f(b)$ are different in sign.

   (c) The Extreme Value Theorem tells us when we can guarantee the existence of maximums and minimums, and tells us where they occur (at endpoints or a critical point).

2. List the three things we need to check to see if a function f is continuous at $x = a$.

   (1) $f(a)$ exists, (2) $\lim_{x\to a} f(x)$ exists, and (3) Parts (1) and (2) have the same value.

3. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

   In general, the distance between a point (a, b) and (x, y) is

   $$d = \sqrt{(x-a)^2 + (y-b)^2}$$

   but minimizing d is equivalent to minimizing

   $$D = (x-a)^2 + (y-b)^2$$

   which is MUCH easier to differentiate. In this case,

   $$D = (x-0)^2 + (y-(-3))^2 = x^2 + (y+3)^2$$

   The equation of the parabola is used to get D in terms of one variable. Since $x + y^2 = 0, x = -y^2$, so

   $$D = (-y^2)^2 + (y+3)^2 = y^4 + (y+3)^2 = y^4 + y^2 + 6y + 9$$

   Now we set the derivative equal to zero and solve,

   $$4y^3 + 2y + 6 = 0$$

   (NOTE: On the exam, I wouldn't give you a cubic equation to solve, in general, these are difficult). HINT: $y = -1$ is one solution.

With this, we can do long division to factor out $(y+1)$:

$$4y^3 + 2y + 6 = 2(y+1)(2y^2 - 2y + 3)$$

From which we get that $y = -1$ is the only critical point (if you use the quadratic formula on $2y^2 - 2y + 3$, you'll get no real solutions).

Now, $x = -(-1)^2 = -1$. Is it a minimum? Yes, which you can get from looking at sign changes of the derivative.

4. Write the equation of the line tangent to $x = \sin(2y)$ at $x = 1$.

   We need a point and a slope. If $x = 1$, then

   $$1 = \sin(2y)$$

   so that $2y = \frac{\pi}{2}$, since $\sin(\pi/2) = 1$. Now, $y = \frac{\pi}{4}$. OK, so now we need a slope:

   $$1 = \cos(2y) \cdot 2\frac{dy}{dx}$$

   and the slope at $y = \frac{\pi}{4}$ is

   $$1 = \cos(\pi/2) \cdot 2\frac{dy}{dx} \Rightarrow 1 = 0$$

   This means that there is no slope- the tangent line is vertical. Therefore, the equation of the tangent line is $x = 1$.

5. The general solution to $(y-k)' = c(y-k)$ is given by:

   $$y - k = Ae^{ct} \qquad y = Ae^{ct} + k$$

   where A could be solved for if we were given a value for $y(0)$.

6. Compute the derivative of y with respect to x:

   (a) $y = \sqrt[3]{2x+1}\sqrt[5]{3x-2}$
   Use the product rule:

   $$y' = \frac{1}{3}(2x+1)^{-2/3} \cdot 2 \cdot \sqrt[5]{3x-2} + \sqrt[3]{2x+1} \cdot \frac{1}{5}(3x-2)^{-4/5} \cdot 3$$

   (b) $y = \frac{1}{1+u^2}$, where $u = \frac{1}{1+x^2}$
   In this case, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, so

   $$\frac{dy}{du} = \frac{-2u}{(1+u^2)^2}, \quad \frac{du}{dx} = \frac{-2x}{(1+x^2)^2}$$

   so

   $$\frac{dy}{dx} = \frac{4ux}{(1+u^2)^2(1+x^2)^2}$$

   with $u = \frac{1}{1+x^2}$, which you can either state or explicitly substitute.

(c) $\sqrt[3]{y} + \sqrt[3]{x} = 4xy$
Implicit differentiation:

$$\frac{1}{3}y^{-2/3}\frac{dy}{dx} + \frac{1}{3}x^{-2/3} = 4y + 4x\frac{dy}{dx}$$

Bring all the $\frac{dy}{dx}$ terms together:

$$\left(\frac{1}{3}y^{-2/3} - 4x\right)\frac{dy}{dx} = 4y - \frac{1}{3}x^{-2/3} \Rightarrow \frac{dy}{dx} = \frac{4y - \frac{1}{3}x^{-2/3}}{\frac{1}{3}y^{-2/3} - 4x}$$

(d) $\sqrt{x+y} = \sqrt[3]{x-y}$
Another implicit differentation:

$$\frac{1}{2}(x+y)^{-1/2}(1+y') = \frac{1}{3}(x-y)^{-2/3}(1-y')$$

Multiply out so that we can isolate y'

$$\frac{1}{2}(x+y)^{-1/2} + y' \cdot \frac{1}{2}(x+y)^{-1/2} = \frac{1}{3}(x-y)^{-2/3} - y' \cdot \frac{1}{3}(x-y)^{-2/3}$$

Now isolate y'

$$y'\left(\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}\right) = \frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}$$

Final answer:

$$y' = \frac{\frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}}$$

(e) $y = \sin(2\cos(3x))$
Chain Rule:

$$y' = \cos(2\cos(3x)) \cdot (-2\sin(3x)) \cdot 3 = -6\cos(2\cos(3x))\sin(3x)$$

(f) $y = (\cos(x))^{2x}$
Logarithmic Differentiation: Write $(\cos(x))^{2x} = \mathrm{e}^{2x\ln(\cos(x))}$, and:

$$y' = \mathrm{e}^{2x\ln(\cos(x))}(\cos(x))^{2x}\left(2\ln(\cos(x)) + 2x\tan(x)\right)$$

(g) $y = (\tan^{-1}(x))^{-1}$
Chain Rule:

$$y' = -(\tan^{-1}(x))^{-2} \cdot \frac{1}{x^2+1}$$

(h) $y = \sin^{-1}(\cos^{-1}(x))$
Chain Rule:

$$y' = \frac{1}{\sqrt{1-(\cos^{-1}(x))^2}} \cdot \frac{-1}{\sqrt{1-x^2}}$$

(i) $y = \log_{10}(x^2 - x)$ $\qquad y' = \frac{1}{(x^2-x)\ln(10)} \cdot (2x-1)$

(j) $y = x^{x^2+2}$ First rewrite: $\mathrm{e}^{(x^2+2)\ln(x)}$, then differentiate:

$$y' = \mathrm{e}^{(x^2+2)\ln(x)}\left(2x\ln(x) + x + \frac{2}{x}\right)$$
$$y' = x^{x^2+2}\left(2x\ln(x) + x + \frac{2}{x}\right)$$

(k) $y = \mathrm{e}^{\cos(x)} + \sin(5^x)$
Chain rule:

$$y' = \mathrm{e}^{\cos(x)}(-\sin(x)) + \cos(5^x)\cdot 5^x\ln(5)$$

(l) $y = \cot(3x^2+5)$
Chain rule:

$$y' = -\csc^2(3x^2+5)\cdot(6x) = -6x\csc^2(3x^2+5)$$

(m) $y = \sqrt{\sin(\sqrt{x})}$
Chain rule:

$$y' = \frac{1}{2}\left(\sin(\sqrt{x})\right)^{-1/2}\cdot\cos(\sqrt{x})\cdot\frac{1}{2}x^{-1/2}$$

(n) $\sqrt{x} + \sqrt[3]{y} = 1$
Implicit Differentiation:

$$\frac{1}{2}x^{-1/2} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{3y^{2/3}}{2x^{1/2}}$$

(o) $x\tan(y) = y - 1$
Product rule/Implicit Diff

$$\tan(y) + x\sec^2(y)y' = y' \Rightarrow \tan(y) = y'(1 - x\sec^2(y))$$

Solve for y':

$$y' = \frac{\tan(y)}{1 - x\sec^2(y)}$$

(p) $y = \frac{-2}{\sqrt[4]{t^3}}$, where $t = \ln(x^2)$.
First, note that $y = -2t^{-3/4}$ and $t = 2\ln(x)$. Now,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

where

$$\frac{dy}{dt} = \frac{-3}{4}t^{-7/4}, \quad \frac{dt}{dx} = \frac{2}{x}$$

so put it all together (and substitute back for x):

$$\frac{dy}{dx} = \frac{-3}{4}\left(2\ln(x)\right)^{-7/4}\cdot\frac{2}{x} = \frac{-6}{4x(2\ln(x))^{7/4}}$$

(q) $y = x3^{-1/x}$

$$y' = 3^{-1/x} + x3^{-1/x}\ln(3)\cdot\frac{1}{x^2} = 3^{-1/x}\left(1+\frac{\ln(3)}{x}\right)$$

7. Find the local maximums and minimums: $f(x) = x^3 - 3x + 1$ Show your answer is correct by using both the first derivative test and the second derivative test.

   To find local maxs and mins, first differentiate to find critical points:

   $$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

   For the first derivative test, set up a sign chart. You should see that $3x^2 - 3 = 3(x+1)(x-1)$ is positive for $x < 1$ and $x > 1$, and $f'(x)$ is negative if $-1 < x < 1$. Therefore, at $x = -1$, the derivative changes sign from positive to negative, so $x = -1$ is the location of a local maximum. At $x = 1$, the derivative changes sign from negative to positive, so we have a local minimum.

   For the second derivative test, we compute the second derivative at the critical points:

   $$f''(x) = 6x$$

   so at $x = -1$, f is concave down, so we have a local max, and at $x = 1$, f is concave up, so we have a local min.

8. Compute the limit, if it exists. You may use any method (except a numerical table).

   (a) $\lim_{x\to 0}\frac{x-\sin(x)}{x^3}$
   We have a form of $\frac{0}{0}$, so use L'Hospital's rule:

   $$\lim_{x\to 0}\frac{x-\sin(x)}{x^3} = \lim_{x\to 0}\frac{1-\cos(x)}{3x^2}$$

   We still have $\frac{0}{0}$, so do it again and again!

   $$\lim_{x\to 0}\frac{1-\cos(x)}{3x^2} = \lim_{x\to 0}\frac{\sin(x)}{6x} = \lim_{x\to 0}\frac{\cos(x)}{6} = \frac{1}{6}$$

   (b) $\lim_{x\to 0}\frac{1-e^{-2x}}{\sec(x)}$
   Note that $\sec(0) = \frac{1}{\cos(0)} = 1$, so this function is continuous at $x = 0$ (we can substitute $x = 0$ in directly), and we get that the limit is 0.

   (c) $\lim_{x\to 4^+}\frac{x-4}{|x-4|}$
   Rewrite the expression to get rid of the absolute value:

   $$\frac{x-4}{|x-4|} = \begin{cases} \frac{x-4}{x-4} & \text{if } x > 4 \\ \frac{x-4}{-(x-4)}, & \text{if } x < 4 \end{cases} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$$

Therefore,

$$\lim_{x\to 4^+} \frac{x-4}{|x-4|} = 1$$

(Note that the overall limit does not exist, however).

(d) $\displaystyle\lim_{x\to-\infty} \sqrt{\frac{2x^2-1}{x+8x^2}}$

For this problem, we should recall that if $x < 0$, then $x = -\sqrt{x^2}$, although in this particular case, the negative signs will cancel:

$$\lim_{x\to-\infty} \sqrt{\frac{2x^2-1}{x+8x^2}} \cdot \frac{\frac{-1}{\sqrt{x^2}}}{\frac{-1}{\sqrt{x^2}}} = \lim_{x\to-\infty} \sqrt{\frac{2-\frac{1}{x^2}}{\frac{1}{x}+8}} = \sqrt{\frac{2}{8}} = \frac{1}{2}$$

(e) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x+1} - \sqrt{x^2-x}$

Multiply by the conjugate (or rationalize):

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - \sqrt{x^2-x} \cdot \frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}.$$

Now divide numerator and denominator:

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = 1$$

(f) $\displaystyle\lim_{h\to 0} \frac{(1+h)^{-2}-1}{h}$

For practice, we'll try it without using L'Hospital's rule:

$$\lim_{h\to 0} \frac{(1+h)^{-2}-1}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2}-1}{h} = \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2} = -2$$

With L'Hospital:

$$\lim_{h\to 0} \frac{(1+h)^{-2}-1}{h} = \lim_{h\to 0} \frac{-2(1+h)^{-3}}{1} = -2$$

EXTRA: This limit was the derivative of some function at some value of x. Name the function and the x value[1].

(g) $\displaystyle\lim_{x\to\infty} x^3 \mathrm{e}^{-x^2}$

First rewrite the function so that it's in an acceptable form for L'Hospital's rule:

$$\lim_{x\to\infty} \frac{x^3}{\mathrm{e}^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2x\mathrm{e}^{x^2}}$$

Note that $\frac{3x^2}{2x\mathrm{e}^{x^2}} = \frac{3x}{2\mathrm{e}^{x^2}}$, and again use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{3x}{2\mathrm{e}^{x^2}} = \lim_{x\to\infty} \frac{3}{4x\mathrm{e}^{x^2}} = 0$$

[1] The function is $f(x) = x^{-2}$ at $x = 1$

(h) $\lim_{x\to 1} \frac{x^{1000}-1}{x-1}$
Using L'Hospital's rule:

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1} = \lim_{x\to 1} \frac{1000x^{999}}{1} = 1000$$

(i) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$
Recall that $\tan^{-1}(0) = 0$, since $\tan(0) = 0$, so this is in a form for L'Hospital's rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$$

(j) $\lim_{x\to 1} x^{\frac{1}{1-x}}$
In this case, recall that $x^{\frac{1}{1-x}} = \mathrm{e}^{\frac{1}{1-x}\cdot\ln(x)}$, so:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} \mathrm{e}^{\frac{1}{1-x}\cdot\ln(x)} = \mathrm{e}^{\lim_{x\to 1} \frac{\ln(x)}{1-x}}$$

so we focus on the exponent:

$$\lim_{x\to 1} \frac{\ln(x)}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

so the overall limit:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \mathrm{e}^{-1}$$

9. Determine all vertical/horizontal asymptotes and critical points of $f(x) = \frac{2x^2}{x^2-x-2}$
The vertical asymptotes: $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$, so $x = -1, x = 2$ are the equations of the vertical asymptotes (note that the numerator is not zero at these values).

The horizontal asymptotes:

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-x-2} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x}-\frac{2}{x^2}} = 2$$

so $y = 2$ is the vertical asymptote (for both $+\infty$ and $-\infty$).

10. Find values of m and b so that (1) f is continuous, and (2) f is differentiable.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 2 \\ mx+b & \text{if} \quad x > 2 \end{cases}$$

First we see that if $x < 2$, $f(x) = x^2$ which is continuous, and if $x < 2$, $f(x) = mx + b$, which is also continuous for any value of m and b. The only problem point is $x = 2$, so we check the three conditions from the definition of continuity:

- $f(2) = 2m + b$, so $f(2)$ exists.
- To compute the limit, we have to do them separately:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} mx + b = 2m + b$$

  For the limit to exist, we must have $4 = 2m + b$. This will also automatically make item 3 true.

  There are an infinite number of possible solutions. Given any m, $b = 4 - 2m$.

For the second part, we know that f must be continuous to be differentiable, so that leaves us with $b = 4 - 2m$. Also, the derivatives need to match up at $x = 2$. On the right side of $x = 2$, $f'(x) = 2x$ and on the left side of $x = 2$, $f'(x) = m$. Therefore, $4 = m$ and $b = 4 - 2 \cdot 4 = -4$.

To be differentiable at $x = 2$, we require $m = 4$ and $b = -4$.

11. Find the local and global extreme values of $f(x) = \frac{x}{x^2+x+1}$ on the interval $[-2, 0]$.

We see that $x^2 + x + 1 = 0$ has no solution, so $f(x)$ is continuous on $[-2, 0]$. Therefore, the extreme value theorem is valid. Next, find the critical points:

$$f'(x) = \frac{(x^2 + x + 1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{-x^2 + 1}{(x^2 + x + 1)^2}$$

so the critical points are $x = \pm 1$ of which we are only concerned with $x = -1$. Now build a chart of values:

x	0	-1	-2
$f(x)$	0	-1	$-2/3$

The minimum occurs at $x = -1$ and the maximum occurs at $x = 0$. The minimum value is -1 and the maximum value is 0.

For the local min/max, use a sign chart for $f'(x)$ in the interval $[-2, 0]$ (or you can use the second derivative test). We see that the denominator of $f'(x)$ is always positive, and the numerator is $1 - x^2$, which changes from negative to positive at $x = -1$, so that f is decreasing then increasing and there is a local minimum at $x = -1$.

12. Suppose f is differentiable so that:

$$f(1) = 1,\ f(2) = 2,\ f'(1) = 1\ f'(2) = 2$$

If $g(x) = f(x^3 + f(x^2))$, evaluate $g'(1)$.

Use the chain rule to get that:

$$g'(x) = f'\left(x^3 + f(x^2)\right) \cdot \left(3x^2 + f'(x^2) \cdot 2x\right)$$

Be careful with the parentheses!:

$$g'(1) = f'(1 + f(1)) \cdot (3 + 2f'(1)) = f'(1 + 1)(3 + 2) = 5f'(2) = 5 \cdot 2 = 10$$

13. Let $x^2y + a^2xy + \lambda y^2 = 0$

(a) Let a and λ be constants, and let y be a function of x. Calculate $\frac{dy}{dx}$:

$$2xy + x^2\frac{dy}{dx} + a^2y + a^2x\frac{dy}{dx} + 2\lambda y\frac{dy}{dx} = 0$$

$$(x^2 + a^2x + 2\lambda y)\frac{dy}{dx} = -(2xy + a^2y) \Rightarrow \frac{dy}{dx} = \frac{-(2xy + a^2y)}{x^2 + a^2x + 2\lambda y}$$

(b) Let x and y be constants, and let a be a function of λ. Calculate $\frac{da}{d\lambda}$:

$$2axy\frac{da}{d\lambda} + y^2 = 0 \Rightarrow \frac{da}{d\lambda} = \frac{-y^2}{2axy}$$

EXTRA[2]: What is $\frac{d\lambda}{da}$?

14. Show that $x^4 + 4x + c = 0$ has at most one solution in the interval $[-1, 1]$.

We don't need the Intermediate Value Theorem here, only the Mean Value Theorem. The derivative is $4x^3 + 4$, so the only critical point is $x = -1$, which is also an endpoint. This implies: (1) If $x^4 + 4x + c = 0$ had two solutions (which is possible), then one of them must be outside the interval, since the two solutions must be on either side of $x = -1$. Therefore, there could be one solution inside the interval. (2) There cannot be any other solution to $x^4 + 4x + c = 0$ inside the interval, because then there would have to be another critical point in $[-1, 1]$. Therefore, we conclude that there is at most one solution inside the interval (there might be no solutions).

15. True or False, and give a short explanation.

(a) If f has an absolute minimum at c, then $f'(c) = 0$.

False. For example, $f(x) = |x|$ has an absolute minimum at $x = 0$, but $f'(x)$ is not defined at $x = 0$.

(b) If $f(x)$ is decreasing and $g(x)$ is decreasing, then $f(x)g(x)$ is decreasing.

False. The derivative is found via the product rule, $f'g + fg'$. We know that $f'(x) < 0$ and $g'(x) < 0$, but we do not know the signs of $f(x)$ and $g(x)$, so we cannot say that the derivative of fg is negative.

(c) If f is differentiable, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$

True, since

$$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

[2]The answer is $\frac{d\lambda}{da} = \frac{2axy}{-y^2}$

(d) $\frac{d}{dx}(10^x) = x10^{x-1}$
False. We cannot use the Power Rule, since there is a variable in the exponent. The correct derivative is found using the rule for a^x:

$$\frac{d}{dx}(10^x) = 10^x \ln(10)$$

(e) If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.
True. If $f'(x)$ exists for all x, then f is differentiable everywhere (and is also continuous everywhere). Thus, the Mean Value Theorem applies. If $f(1) = f(0)$, that would imply the existence of a c in the interval $(0, 1)$ so that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0$$

but we're told that $f'(x) \neq 0$.

(f) If $y = ax + b$, then $\frac{dy}{da} = x$
True. If we're computing $\frac{dy}{da}$, then we're treating x and b as constants. Differentiating, we get

$$\frac{dy}{da} = x + 0 = x$$

(g) If $2x + 1 \leq f(x) \leq x^2 + 2$ for all x, then $\lim_{x\to 1} f(x) = 3$.
True. This is the Squeeze Theorem. If $f(x)$ is trapped between $2x + 1$ and $x^2 + 2$ for all x, and since the limit as $x \to 1$ of $2x + 1$ is 3, and the limit as $x \to 1$ of $x^2 + 2 = 3$, then that forces the limit as $x \to 1$ of $f(x)$ to also be 3.

(h) If $f'(r)$ exists, then

$$\lim_{x\to r} f(x) = f(r)$$

True. The statement that $f'(r)$ exists says that f is differentiable at r. The statement that $\lim_{x\to r} f(x) = f(r)$ is asking if f is continuous at r. We know that all differentiable functions are continuous, so the statement is True.

(i) If f and g are differentiable, then:

$$\frac{d}{\mathrm{dx}}(f(g(x)) = f'(x)g'(x)$$

False. The chain rule states that $\frac{d}{\mathrm{dx}}(f(g(x)) = f'(g(x))g'(x)$

(j) If $f(x) = x^2$, then the equation of the tangent line at $x = 3$ is: $y - 9 = 2x(x - 3)$
False. $2x$ is a formula for the slope, not the slope itself. The equation of the tangent line is: $y - 9 = 6(x - 3)$.

(k) $\lim_{\theta\to\frac{\pi}{3}} \dfrac{\cos(\theta) - \frac{1}{2}}{\theta - \frac{\pi}{3}} = -\sin\left(\dfrac{\pi}{3}\right)$
True by l'Hospital's rule. You could have also said that the expression is in the form:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

with $f(x) = \cos(x)$ and $a = \frac{\pi}{3}$. This is another way of defining the derivative of $\cos(x)$ at $x = \frac{\pi}{3}$.

(l) There is no solution to $e^x = 0$

True. If there were a solution, it would be $x = \ln(0)$, but $\ln(0)$ is not defined.

(m) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

False, with the usual restrictions on the sine function. That is, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then it is true that $\sin^{-1}(\sin(\theta)) = \theta$. In this case,

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$$

(n) $5^{\log_5(2x)} = 2x$, for $x > 0$.

True, since 5^x and $\log_5(x)$ are inverses of each other. We needed $x > 0$ so that $\log_5(2x)$ is defined.

(o) $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$, for all $x \neq 0$.

True:

$$\ln|x| = \begin{cases} \ln(x), & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

so

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

(p) $\frac{d}{dx}10^x = x10^{x-1}$

Oops- a duplication. See part (c)

(q) If $x > 0$, then $(\ln(x))^6 = 6\ln(x)$

False. The rule says: $\log(a^b) = b\log(a)$, but here the 6 is outside the logarithm.

16. Find the domain of $\ln(x - x^2)$:

Use a sign chart to determine where $x - x^2 = x(1-x) > 0$:

x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
	$x < 0$	$0 < x < 1$	$x > 1$

so overall, $x - x^2 > 0$ if $0 < x < 1$.

17. Find the value of c guaranteed by the Mean Value Theorem, if $f(x) = \frac{x}{x+2}$ on the interval $[1, 4]$.

To set things up, we see that f is continuous on $[1, 4]$ and differentiable on $(1, 4)$, since the only "bad point" is $x = -2$. We should get that $f'(x) = \frac{2}{(x+2)^2}$, $f(1) = \frac{1}{3}$ and $f(4) = \frac{2}{3}$. Therefore, the Mean Value Theorem says that c should satisfy:

$$\frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{1}{9}$$

or

$$(c+2)^2 = 18 \Rightarrow c = -2 \pm \sqrt{18}$$

of which only $-2+\sqrt{18} \approx 2.243$ is inside our interval.

18. Given that the graph of f passes through the point $(1,6)$ and the slope of the tangent line at $(x, f(x))$ is $2x+1$, find $f(2)$.

    Since $f'(x) = 2x+1$, $f(x) = x^2+x+C$ is the general antiderivative. Given that $(1,6)$ goes through f, $1^2+1+C = 6 \Rightarrow C = 4$. Therefore, $f(x) = x^2+x+4$. Now, $f(2) = 4+2+4 = 10$.

19. To simplify, we make $\theta = \tan^{-1}(x)$ so that $\tan(\theta) = \frac{x}{1}$. This means we are looking at a triangle whose length of the side opposite is x, and adjacent is 1. The hypotenuse is then $\sqrt{1+x^2}$. Now take the $\cos(\theta)$ to get:

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

20. A fly is crawling from left to right along the curve $y = 8-x^2$, and a spider is sitting at $(4,0)$. At what point along the curve does the spider first see the fly?

    Another way to say this: What are the tangent lines through $y = 8-x^2$ that also go through $(4,0)$?

    The unknown value here is the $x-$coordinate, so let $x = a$. Then the slope is $-2a$, and the corresponding point on the curve is $(a, 8-a^2)$. The general form of the equation of the tangent line is then given by:

$$y - 8 + a^2 = -2a(x-a)$$

where x, y are points on the tangent line. We want the tangent line to go through $(4,0)$, so we put this point in and solve for a:

$$-8+a^2 = -2a(4-a) = -8a+2a^2 \quad \Rightarrow \quad 0 = a^2-8a+8$$

$$\Rightarrow \quad a = \frac{8 \pm \sqrt{32}}{2}$$

so we take the leftmost value, $a = \frac{8-\sqrt{32}}{2}$.

21. Compute the limit, without using L'Hospital's Rule. $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

    Rationalize to get:

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{1}{(\sqrt{x+2}+3)} = \frac{1}{6}$$

which is the derivative of $\sqrt{x+2}$ at $x = 7$.

22. For what value(s) of c does $f(x) = cx^4 - 2x^2 + 1$ have both a local maximum and a local minimum?

First, $f'(x) = 4cx^3 - 4x = 4x(cx^2 - 1)$, and $f''(x) = 12cx^2 - 4$. The candidates for the location of the local max's and min's are where $f'(x) = 0$, which are $x = 0$ and $x = \pm\sqrt{1/c}$ ($c > 0$). We can use the second derivative test to check these out:

At $x = 0$, $f''(0) = -4$, so $x = 0$ is always a local max. At $x = \pm\sqrt{1/c}$, $f''(\pm\sqrt{1/c}) = 12 - 4 = 8$. So, if $c > 0$, there are local mins at $x = \pm\sqrt{1/c}$.

23. If $f(x) = \sqrt{1 - 2x}$, determine $f'(x)$ *by using the definition of the derivative.*

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2(x+h)} + \sqrt{1-2x}}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} =$$

$$\lim_{h\to 0} \frac{1 - 2x - 2h - 1 + 2x}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} =$$

$$\lim_{h\to 0} \frac{-2}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} = \frac{-1}{\sqrt{1-2x}}$$

24. A *point of inflection* for a function f is the x value for which $f''(x)$ changes sign (either from positive to negative or vice versa).

> If f'' is continuous, and $f''(a) < 0$ and $f''(b) > 0$, there is a c so that $f''(c) = 0$ is an inflection point.

...which is the Intermediate Value Theorem.

Find constants a and b so that $(1, 6)$ is an inflection point for $y = x^3 + ax^2 + bx + 1$.

Differentiate twice to get:

$$y'' = 6x + 2a$$

At $x = 1$, we want an inflection point, so $6 + 2a$ should be a point where y'' changes sign: $6 + 2a = 0 \Rightarrow a = -3$. We see that if $a < -3$, then $y'' < 0$, and if $a > -3$, $y'' > 0$.

Putting this back into the function, we have:

$$y = x^3 - 3x^2 + bx + 1$$

and we want the curve to go through the point $(1, 6)$:

$$6 = 1 - 3 + b + 1$$

so $b = 7$.

25. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f(3) = 2$ and $f'(6) = 7$. Find $F'(3)$.

    By the Chain Rule:

    $$F'(3) = f'(g(3))g'(3)$$

    so $F'(3) = f'(6) \cdot 4 = 7 \cdot 4 = 28$

26. Find the dimensions of the rectangle of largest area that has its base on the $x-$axis and the other two vertices on the parabola $y = 8 - x^2$.

    Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$.

    Now, let x be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

    $$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

    and $0 \leq x \leq \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

    $$\frac{dA}{dx} = 16 - 6x^2$$

    so the critical points are $x = \pm\frac{4}{\sqrt{6}}$, of which only $x = \frac{4}{\sqrt{6}}$ is in our interval. So the dimensions of the rectangle are:

    $$2x = \frac{8}{\sqrt{6}}, y = 5\frac{1}{3}$$

27. Let $G(x) = h(\sqrt{x})$. Then where is G differentiable? Find $G'(x)$.

    First compute $G'(x) = h'(\sqrt{x})\frac{1}{2}x^{-1/2}$. From this we see that as long as h is differentiable and $x > 0$, then G will be differentiable.

28. If position is given by: $f(t) = t^4 - 2t^3 + 2$, find the times when the acceleration is zero. Then compute the velocity at these times.

    Take the second derivative, and set it equal to zero:

    $$f'(x) = 4t^3 - 6t^2, \quad f''(t) = 12t^2 - 12t = 0 \Rightarrow t = 0, t = 1$$

    The velocity at $t = 0$ is 0 and the velocity at $t = 1$ is $4 - 6 = -2$.

29. If $y = \sqrt{5t - 1}$, compute y'''.

    Nothing tricky here- Just differentiate, and differentiate, and differentiate!

    $$y' = \frac{1}{2}(5t-1)^{-1/2} \cdot 5 = \frac{5}{2}(5t-1)^{-1/2}$$

    $$y'' = \frac{5}{2} \cdot \frac{-1}{2}(5t-1)^{-3/2} \cdot 5 = \frac{-25}{4}(5t-1)^{-3/2}$$

    $$y''' = \frac{375}{8}(5t-1)^{-5/2}$$

30. If $f(x) = (2-3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$.

Differentiate:

$$f'(x) = \frac{-1}{2}(2-3x)^{-3/2}(-3) = \frac{3}{2}(2-3x)^{-3/2}$$
$$f''(x) = \frac{-9}{4}(2-3x)^{-5/2}(-3) = \frac{27}{4}(2-3x)^{-5/2}$$

Now, (note that $2^{3/2} = 2\sqrt{2}$ and $2^{5/2} = 4\sqrt{2}$):

$$f(0) = \frac{1}{\sqrt{2}}, \quad f'(0) = \frac{3}{2}\cdot\frac{1}{2^{3/2}} = \frac{3}{4\sqrt{2}}, \quad f''(0) = \frac{27}{16\sqrt{2}}$$

31. Car A is traveling west at 50 mi/h, and car B is traveling north at 60 mi/h. Both are headed for the intersection between the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Let $A(t)$, $B(t)$ be the positions of cars A and B at time t. Let the distance between them be $z(t)$, so that the Pythagorean Theorem gives:

$$z^2 = A^2 + B^2$$

Translating the question, we get that we want to find $\frac{dz}{dt}$ when $A = 0.3$, $B = 0.4$, (so $z = 0.5$), $A'(t) = 50$, $B'(t) = 60$. Then:

$$2z\frac{dz}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}$$

The two's divide out and put in the numbers:

$$0.5\cdot\frac{dz}{dt} = 0.3\cdot 50 + 0.4\cdot 60$$

and solve for $\frac{dz}{dt}$, 78.

32. Find the linearization of $f(x) = \sqrt{1-x}$ at $x = 0$.

To linearize, we find the equation of the tangent line.

$$f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1)$$

so $f'(0) = -\frac{1}{2}$, and the point is $(0,1)$.

$$y - 1 = -\frac{1}{2}x, \quad \text{or} \quad y = -\frac{1}{2} + 1$$

33. Find $f(t)$, if $f''(t) = t + \sqrt{t}$, and $f(1) = 1$, $f'(1) = 2$.

$$f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + C$$

so $f'(1) = 2$ means:

$$\frac{1}{2} + \frac{2}{3} + C = 2, \text{ so } C = \frac{5}{6}$$

Now, $f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + \frac{5}{6}$, and

$$f(t) = \frac{1}{2} \cdot \frac{1}{3}t^3 + \frac{2}{3} \cdot \frac{2}{5}t^{5/2} + \frac{5}{6}t + C = \frac{1}{6}t^3 + \frac{4}{15}t^{5/2} + \frac{5}{6}t + C$$

Now, $f(1) = 1$ means:

$$\frac{1}{6} + \frac{4}{15} + \frac{5}{6} + C = 1 \Rightarrow \frac{5+8+25}{30} + C = 1 \Rightarrow C = 1 - \frac{19}{15} = \frac{-4}{15}$$

34. Find $f'(x)$ directly from the definition of the derivative (using limits and without L'Hospital's rule):

First, recall that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We'll use this for these exercises:

(a) $f(x) = \sqrt{3 - 5x}$

$$\lim_{h \to 0} \frac{\sqrt{3-5x-5h} - \sqrt{3-5x}}{h} \cdot \frac{\sqrt{3-5x-5h} + \sqrt{3-5x}}{\sqrt{3-5x-5h} + \sqrt{3-5x}}$$

$$\lim_{h \to 0} \frac{3 - 5x - 5h - 3 + 5x}{h\left(\sqrt{3-5x-5h} + \sqrt{3-5x}\right)}$$

$$\frac{-5}{2\sqrt{3-5x}}$$

(b) $f(x) = x^2$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$\lim_{h \to 0} \frac{h(2x+h)}{h} = 2x$$

(c) $f(x) = x^{-1}$

$$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} =$$

$$\lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

35. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

First, note that the derivative is (Chain Rule):

$$f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

which simplifies (since $f(0) = 0$) to:

$$f'(0) \cdot f'(0) \cdot f'(0) \cdot f'(0) = 2^4 = 16$$

36. Differentiate:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{x} & \text{if } x < 0 \end{cases}$$

Is f differentiable at $x = 0$? Explain.

Correction: $-\sqrt{x} = -\sqrt{-x}$

Is f differentiable at $x = 0$? Explain.

f will not be differentiable at $x = 0$. Note that, if $x > 0$, then $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(x) \to \infty$ as $x \to 0^+$

If $x < 0$, $f'(x) = \frac{1}{2\sqrt{-x}}$, which also goes to infinity as x approaches 0 (from the left).

37. $f(x) = |\ln(x)|$. Find $f'(x)$.

We can rewrite f (Recall that $\ln(x) < 0$ if $0 < x < 1$)

$$f(x) = \begin{cases} \ln(x), & \text{if } x \geq 1 \\ -\ln(x), & \text{if } 0 < x < 1 \end{cases}$$

and differentiate piecewise:

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 1 \\ -\frac{1}{x}, & \text{if } 0 < x < 1 \end{cases}$$

Note that the pieces don't match at $x = 1$; we remove that point from the domain.

38. $f(x) = xe^{g(\sqrt{x})}$. Find $f'(x)$.

$$f'(x) = e^{g(\sqrt{x})} + xe^{g(\sqrt{x})} \cdot g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

39. Find a formula for dy/dx: $x^2 + xy + y^3 = 0$.

$$2x + y + xy' + 3y^2y' = 0 \Rightarrow (x + 3y^2)y' = -(2x + y) \Rightarrow y' = \frac{-(2x + y)}{x + 3y^2}$$

40. Show that 5 is a critical number of $g(x) = 2 + (x - 5)^3$, but that g does not have a local extremum there.

$g'(x) = 3(x - 5)^2$, so $g'(5) = 0$.

By looking at the sign of $g'(x)$ (First derivative test), we see that $g'(x)$ is always non-negative, so g does not have a local min or max at $x = 5$.

41. Find the general antiderivative: (SORRY, THIS SHOULD NOT HAVE BEEN INCLUDED HERE- IGNORE THIS PROBLEM FOR NOW)

    (a) $f(x) = 4 - x^2 + 3e^x$ $F(x) = 4x - \frac{1}{3}x^3 + 3e^x + C$

    (b) $f(x) = \frac{3}{x^2} + \frac{2}{x} + 1$ $F(x) = -3x^{-1} + 2\ln|x| + x + C$

(c) $f(x) = \frac{1+x}{\sqrt{x}}$

First rewrite $f(x) = x^{-1/2} + x^{1/2}$, and

$$F(x) = 2x^{1/2} + \frac{2}{3}x^{3/2} + C$$

42. Find the slope of the tangent line to the following at the point (3,4): $x^2 + \sqrt{y}x + y^2 = 31$

$$2x + \frac{1}{2}y^{-1/2}y'x + \sqrt{y} + 2yy' = 0$$

At $x = 3, y = 4$:

$$6 + \frac{3}{4}y' + 2 + 8y' = 0 \quad \Rightarrow \quad y' = \frac{-32}{35}$$

$$y - 4 = \frac{-32}{35}(x - 3)$$

43. Find the critical values: $f(x) = |x^2 - x|$

One way to approach this problem is to look at it piecewise. Use a table to find where $f(x) = x(x-1)$ is positive or negative:

$$f(x) = \begin{cases} x^2 - x & \text{if } x \le 0, \text{ or } x \ge 1 \\ -x^2 + x & \text{if } 0 < x < 1 \end{cases}$$

Now compute the derivative:

$$f'(x) = \begin{cases} 2x - 1 & \text{if } x < 0, \text{ or } x > 1 \\ -2x + 1 & \text{if } 0 < x < 1 \end{cases}$$

At $x = 0$, from the left, $f'(x) \to 1$ and from the right, $f'(x) \to -1$, so $f'(x)$ does not exist at $x = 0$.

At $x = 1$, from the left, $f'(x) \to -1$, and from the right, $f'(x) \to 1$, so $f'(x)$ does not exist at $x = 1$.

Finally, $f'(x) = 0$ if $2x - 1 = 0 \Rightarrow x = 1/2$, but $1/2$ is not in that domain. The other part is where $-2x + 1 = 0$, which again is $1/2$, and this time it is in $0 < x < 1$.

The critical points are: $x = 1/2, 0, 1$.

44. Does there exist a function f so that $f(0) = -1$, $f(2) = 4$, and $f'(x) \le 2$ for all x?

$$f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}$$

Since $\frac{5}{2} > 2$, there can exist no function like that (that is continuous).

45. Linearize $f(x) = \sqrt{1 + x}$ at $x = 0$.

Point: $x = 0, y = 1$

Slope: $f'(0) = \frac{1}{2}$

Line: $y - 1 = \frac{1}{2}(x - 0)$, or $y = \frac{1}{2}x + 1$

46. (THIS PROBLEM SHOULD HAVE BEEN DELETED- SORRY!) Find dy if $y = \sqrt{1-x}$ and evaluate dy if $x = 0$ and $dx = 0.02$. Compare your answer to Δy

$$dy = \frac{-1}{2\sqrt{1-x}}\,dx, \quad \Rightarrow dy = \frac{1}{2\sqrt{1-0}} \cdot 0.02 = 0.01$$

$$\Delta y = \sqrt{1-0.02} - \sqrt{1} = 0.01005...$$

47. Fill in the question marks: If f'' is positive on an interval, then f' is INCREASING and f is CONCAVE UP.

48. If $f(x) = x - \cos(x)$, x is in $[0, 2\pi]$, then find the value(s) of x for which

    (a) $f(x)$ is greatest and least.
    Here we are looking for the maximum and minimum- use a table with endpoints and critical points. To find the critical points,

    $$f'(x) = 1 + \sin(x) = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

    is the only critical point in $[0, 2\pi]$.
    Now the table:

x	0	$\frac{3\pi}{2}$	2π
$f(x)$	-1	$\frac{3}{\pi}2 \approx 4.7$	$2\pi - 1 \approx 5.2$

    so f is greatest at $x = 2\pi$, least at 0.

    (b) $f(x)$ is increasing most rapidly.
    Another way to say this: Where's the maximum of $f'(x)$? We've computed $f'(x)$ to be: $1+\sin(x)$, so take its derivative: $\cos(x) = 0$. So there are two critical points at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Checking these and the endpoints:

x	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
$f'(x)$	1	2	0	1

    so f is increasing most rapidly at $x = \frac{\pi}{2}$.

    (c) The slopes of the lines tangent to the graph of f are increasing most rapidly.
    Another way to say this: Where is $f'(x)$ increasing most rapidly? At the maximum of $f''(x)$. The maximum of $\cos(x)$ in the interval $[0, 2\pi]$ occurs at $x = 0$ and $x = 2\pi$.

49. Show there is *exactly* one root to: $\ln(x) = 3 - x$

    First, to use the Intermediate Value Theorem, we'll get a function that we can set to zero: Let $f(x) = \ln(x) - 3 + x$. Then a root to $\ln(x) = 3 - x$ is where $f(x) = 0$.

    First, by plugging in numbers, we see that $f(2) < 0$ and $f(3) > 0$. There is at least one solution in the interval $[2, 3]$ by the Intermediate Value Theorem.

    Now, is there more than one solution? $f'(x) = \frac{1}{x} + 1$ which is always positive for positive x. This means that, for $x > 0$, $f(x)$ is always increasing. Therefore, if it crosses the $x-$axis (and it does), then f can never cross again.

50. Sketch the graph of a function that satisfies all of the given conditions:

$$\begin{array}{ccc} f(1)=5 & f(4)=2 & f'(1)=f'(4)=0 \\ \lim_{x\to 2^+} f(x)=\infty, & \lim_{x\to 2^-} f(x)=3 & f(2)=4 \end{array}$$

(We'll do this one in class)

51. If $s^2t+t^3=1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.
First, treat t as the function, s as the variable:

$$2st+s^2\frac{dt}{ds}+3t^2\frac{dt}{ds}=0 \Rightarrow \frac{dt}{ds}=\frac{-2st}{s^2+3t^2}$$

For s as the function, t as the variable:

$$\frac{ds}{dt}=\frac{-(s^2+3t^2)}{2st}$$

which you can either state directly or show.

52. Define the functions as piecwise defined functions:

$$\frac{|3x+2|}{3x+2}=\begin{cases} \frac{3x+2}{3x+2} & \text{if } 3x+2>0 \\ -\frac{3x+2}{3x+2} & \text{if } 3x+2<0 \end{cases} = \begin{cases} 1 & \text{if } x>-\frac{2}{3} \\ -1 & \text{if } x<-\frac{2}{3} \end{cases}$$

To do the next problem, we need to know where the function is positive and where it is negative. Use a sign chart:

$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<2$	$x>2$

The function is positive in the second and 4th intervals, and negative in the first and third. Now,

$$\left|\frac{x-2}{(x+1)(x+2)}\right| = \begin{cases} \frac{x-2}{(x+1)(x+2)} & \text{if } -2<x<-1 \text{ or } x\geq 2 \\ -\frac{x-2}{(x+1)(x+2)} & \text{if } x<-2 \text{ or } -1<x<2 \end{cases}$$

53. Find all values of c and d so that f is continuous at all real numbers.

First, we should see that the only two "problem" points are at $x=0$ and at $x=1$, which are where the functions come together. Other than those points, f is continuous. At the problem points, we need to check the three conditions for continuity.

At $x=0$, (i) $f(0)=c\cdot 0+d$, so $f(0)=d$ which exists for all c and d. (ii) Check that the limit exists:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} 2x^2-1=-1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} cx + d = d$$

Therefore, for the limit to exist at $x = 0$, we must have $d = -1$. So far, this function is now continuous at $x = 0$ for any value of c.

At $x = 1$, (i) $f(1) = c \cdot 1 + d = c \cdot 1 - 1 = c - 1$, which exists for all values of c. (ii) Check that the limit exists at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} cx - 1 = c - 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x+3} = 2$$

Therefore, for the limit to exist at $x = 1$, $c - 1 = 2$ or $c = 3$. At this value of c, the third condition of continuity is also satisfied.

Final answer: $c = 3, d = -1$.

54. Where is f continuous? Use a sign chart, since the expression under the radical sign must be nonnegative (positive or zero):

$x-1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
	$x < -2$	$-2 < x < 1$	$1 < x < 2$	$x > 2$

Therefore, $f(x) = \sqrt{\frac{x-1}{x^2-4}}$ is continuous on its domain, which is:

$$-2 < x \le 1 \text{ or } x > 2$$

55. Recall our setup- we want to let $Q(t)$ be the kilograms of salt at time t (NOT the concentration- that would require a different equation). The differential equation is then:

$$\frac{dQ}{dt} = \text{Rate coming in} - \text{Rate going out}$$

The rate coming in is

$$25\frac{\text{liters}}{\text{minute}} \cdot 0.03\frac{\text{kg}}{\text{liter}} = 0.75\frac{\text{kg}}{\text{min}}$$

To compute the rate going out, there is Q kilograms of salt in 5000 liters of water, so $Q/5000$ is the concentration at time t. Therefore, the rate of salt leaving the tank is:

$$25\frac{\text{liters}}{\text{min}} \cdot \frac{Q}{5000}\frac{\text{kg}}{\text{liters}} = 0.005Q\frac{\text{kg}}{\text{min}}$$

Now we have:

$$Q' = 0.75 - 0.005Q = -0.005\left(Q - \frac{0.75}{0.005}\right) = -0.005(Q - 150)$$

which we write as:

$$(Q - 150)' = -0.005(Q - 150) \Rightarrow Q(t) = Ae^{-0.005t} + 150$$

We solve for A by noting that $Q(0) = 20$, so that

$$Q(t) = -130\mathrm{e}^{-0.005t} + 150$$

Now, after a half an hour ($t = 30$), we get $Q(30) \approx 38.11$.

56. It might be interesting to compare the results of this problem if we assume that (i) the vertex is up, versus (ii) the vertex is down. (An inverted cone has its vertex up, so only (i) would be the solution to the problem as given):

    (a) In this case, let r be the radius on the surface of the water level at height h. Then the volume of water in the cone is given by the full volume, and subtract the empty cone on the top:

    $$V = \frac{\pi}{3}\pi \cdot 2^2 \cdot 4 - \frac{\pi}{3}\pi r^2 h$$

    Now we need to write V in terms of h, since we are trying to determine $\frac{dh}{dt}$. From similar triangles, we get:

    $$\frac{2}{4} = \frac{r}{4-h} \qquad r = \frac{4-h}{2}$$

    Now the volume becomes

    $$V = \frac{16\pi}{3} - \frac{\pi}{12}(h^3 - 8h^2 + 16h) \qquad \frac{dV}{dt} = -\frac{\pi}{12} \cdot (3h^2 - 16h + 16)\frac{dh}{dt}$$

    Now substitute $\frac{dV}{dt} = 2$, $h = 3$ to find that $\frac{dh}{dt} = \frac{24}{5\pi}$

    (b) If the cone had its vertex at the bottom, the problem simplifies quite a bit. The volume of water is just the volume of the small cone,

    $$V = \frac{\pi}{3} r^2 h$$

    where by similar triangles we get $h = 2r$. Substituting like we did earlier,

    $$V = \frac{\pi}{12} h^3 \qquad \frac{dV}{dt} = \frac{\pi}{4} h^2 \cdot \frac{dh}{dt}$$

    Substitute $\frac{dV}{dt} = 2$ and $h = 3$ to get $\frac{dh}{dt} = \frac{8}{9\pi}$

57. We did something similar to this problem in class. Form a right triangle where the hypotenuse is 25. The other two sides have length $x(t)$ and $y(t)$ (for specificity, let $x(t)$ be the distance from the base of the wall to the base of the ladder, and $y(t)$ be the height of the ladder against the wall). Then,

    $$x^2 + y^2 = 625 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

    Now, we are given that $\frac{dy}{dt} = -1$ and $x = 6$, so we still need y in order to solve for $\frac{dx}{dt}$. From our original equation, if $x = 6$, then

    $$y^2 = 25^2 - 6^2 = 589 \Rightarrow y = \sqrt{589}$$

Now substitute these values in, and:

$$12\frac{dx}{dt} - 2\sqrt{589} = 0 \Rightarrow \frac{dx}{dt} = \frac{\sqrt{589}}{6}$$

58. The snowball equation is:

$$A = 4\pi r^2$$

where A and r are functions of t. We need to write A in terms of the diameter instead of in terms of the radius. Let h be the diameter, $h = 2r$ and substitute:

$$A = 4\pi\frac{h^2}{4} = \pi h^2$$

Differentiating with respect to time, we get:

$$\frac{dA}{dt} = 2\pi h\frac{dh}{dt}$$

We are told that $\frac{dA}{dt} = -1$, $h = 10$ and solve for $\frac{dh}{dt}$,

$$\frac{dh}{dt} = -\frac{1}{20\pi} \approx -0.0159$$

59. For this tank mixing problem, we have:

$$\frac{dQ}{dt} = 10 \cdot 0 - 10 \cdot \frac{Q}{1000} = -\frac{1}{100}Q$$

so the general solution is $Q(t) = A\mathrm{e}^{-\frac{1}{100}t}$. Using the initial condition that $Q(0) = 15$, we get:

$$Q(t) = 15\mathrm{e}^{-\frac{1}{100}t}$$

The mathematical model of Q says that there will never be zero salt in any finite amount of time (although in practice, the amount is negligible).