Solutions to Exercises

1. Evaluate $\lim\limits_{x \to 4} \dfrac{x^2 - x - 12}{x - 4}$

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x \to 4} \frac{(x+3)(x-4)}{x - 4} = 4 + 3 = 7$$

2. Evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 5x^2 + 2x - 4}{x^2 - 3x + 3}$ Everything is defined; evaluate at $x = 2$.

$$\lim_{x \to 2} \frac{x^3 - 5x^2 + 2x - 4}{x^2 - 3x + 3} = -12$$

3. Find $\lim\limits_{x \to \infty} \dfrac{2x + 5}{x^2 - 7x + 3}$

   Multiply numerator and denominator by $\frac{1}{x^2}$, getting

$$\lim_{x \to \infty} \frac{\frac{2}{x} + \frac{5}{x^2}}{1 - \frac{7}{x} + \frac{3}{x^2}} = \frac{0}{1} = 0$$

4. Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{x+3} - \sqrt{3}}{x}$ Rationalize numerator and denominator:

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} = \lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \frac{\sqrt{x+3} + \sqrt{3}}{\sqrt{x+3} + \sqrt{3}} = \frac{(x+3) - 3}{x(\sqrt{x+3} + \sqrt{3})} = \lim_{x \to 0} \frac{1}{\sqrt{x+3} + \sqrt{3}} = \frac{1}{2\sqrt{3}}$$

5. Evaluate $\lim\limits_{x \to -\infty} (2x^3 - 12x^2 + x - 7)$ The leading term will determine what happens: As $x \to -\infty$, the function goes to $-\infty$.

6. Evaluate $\lim\limits_{x \to 2} \left(\dfrac{1}{x - 2} - \dfrac{4}{x^2 - 4} \right)$

   Simplify the expression before attempting the limit:

$$\frac{1}{x-2} - \frac{4}{x^2 - 4} = \frac{(x+2) - 4}{x^2 - 4} = \frac{x - 2}{(x+2)(x-2)} = \frac{1}{x+2}$$

   Now, as $x \to 2$, the expression goes to $\frac{1}{4}$.

7. Evaluate $\lim\limits_{x \to 3} \dfrac{1}{(x-3)^2}$ This is a shifted version of: $\lim\limits_{x \to 0} \dfrac{1}{x^2}$, which is ∞.

8. Find $\lim\limits_{x \to \infty} \dfrac{4x - 1}{\sqrt{x^2 + 2}}$

   Multiply numerator and denominator by $\frac{1}{x}$- For the denominator, this means $\frac{1}{\sqrt{x^2}}$:

$$\lim_{x \to \infty} \frac{4 - \frac{1}{x}}{\sqrt{1 + \frac{2}{x^2}}} = 4$$

9. Find $\lim\limits_{x \to -\infty} \dfrac{4x - 1}{\sqrt{x^2 + 2}}$

   Same idea as the last problem, but since $x < 0$, $x = -\sqrt{x^2}$. Multiply the numerator by $\frac{1}{x}$ and the denominator by $-\frac{1}{\sqrt{x^2}}$ (keep the negative sign out front):

$$\lim_{x \to \infty} \frac{4 - \frac{1}{x}}{-\sqrt{1 + \frac{2}{x^2}}} = -4$$

10. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ if:

$$f(x) = \begin{cases} 7x-2, & \text{if } \ x \geq 2 \\ 3x+5, & \text{if } \ x < 2 \end{cases}$$

$\lim_{x\to 2^+} f(x)$ is found by using the top function, since $x > 2$. The limit is $7(2) - 2 = 12$.
$\lim_{x\to 2^-} f(x)$ is found by using the bottom function, since $x < 2$. The limit is $3(2) + 5 = 11$.
Because the limits from the right and left are not the same, the function is not continuous at $x = 2$.

11. Find $\lim_{x\to 2} \dfrac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x}$

Rationalize and simplify, then take the limit:

$$\frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x} \cdot \frac{\sqrt{x+2} + \sqrt{2x}}{\sqrt{x+2} + \sqrt{2x}} = \frac{x+2-2x}{x(x-2)(\sqrt{x+2} + \sqrt{2x})} = \frac{-1}{x(\sqrt{x+2} + \sqrt{2x})}$$

so the limit is: $\frac{-1}{2(\sqrt{4}+\sqrt{4})} = \frac{-1}{8}$

12. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ if $f(x) = \dfrac{|x-2|}{x-2}$

We see that: $|x-2| = x-2$ if $x > 2$ and $|x-2| = -(x-2)$ if $x < 2$. Therefore, the right limit is 1 and the left limit is -1.

13. Find $\lim_{x\to\infty} \tan^{-1}(2x+1)$

As $x \to \infty$, then $2x+1 \to \infty$. If the input to $\tan^{-1}(x)$ goes to positive infinity, then $\tan^{-1}(x)$ approaches its horizontal asymptote at $y = \frac{\pi}{2}$, so the limit is $\frac{\pi}{2}$.

14. Find $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, if $f(x) = x^2 - 4x$.

First get the expression requested and simplify before taking the limit:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 4(x+h) - [x^2 - 4x]}{h} = \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h} = 2x + h - 4$$

Now, take the limit to get the answer of $2x - 4$.

15. Find all vertical and horizontal asymptotes for $\sqrt{x+1} - \sqrt{x}$

There are no vertical asymptotes, and the domain is $x \geq 0$, so we only (perhaps) have a horizontal asymptote:

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} = \lim_{x\to\infty} \frac{(x+1) - x}{\sqrt{x+1} + \sqrt{x}} = 0$$

16. Find all vertical and horizontal asymptotes for $\frac{x^2-5x+6}{x-3}$

First, factor the numerator to see if we can simplify:

$$\frac{x^2 - 5x + 6}{x-3} = \frac{(x-3)(x+2)}{x-3} = x + 2, \text{ if } \ x \neq 3$$

We see that there are no vertical or horizontal asymptotes.

17. Find all vertical and horizontal asymptotes for $\frac{2x+3}{\sqrt{x^2-2x-3}}$

First, factor out the denominator: $x^2 - 2x - 3 = (x-3)(x+1)$. Since these do not cancel with the numerator, there are vertical asymptotes at $x = 3$ and $x = -1$. Now for the horizontal asymptotes:

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2 - 2x - 3}} = \lim_{x\to\infty} \frac{2 + \frac{3}{x}}{\sqrt{1 - \frac{2}{x} - \frac{3}{x^2}}} = 2$$

Note that we used the fact that, if $x > 0$, then $x = \sqrt{x^2}$. If we take $x \to -\infty$, we use the fact that $x = -\sqrt{x^2}$:

$$\lim_{x\to-\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{-\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}} = -2$$

18. Evaluate $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x}$

$$\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x} = \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x(x-2)} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x(x-2)(\sqrt{x^2+5}+3)} = \frac{1}{3}$$

19. Evaluate $\lim_{x\to\infty} (\sqrt{x^2+2x}-x)$

$$\lim_{x\to\infty} (\sqrt{x^2+2x}-x) \cdot \frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = 1$$

20. Find $\lim_{x\to\pm\infty} \frac{7x^3+2x^2}{4x^3-x} = \lim_{x\to\pm\infty} \frac{7+\frac{2}{x}}{4-\frac{1}{x^2}} = \frac{7}{4}$

21. Find $\lim_{x\to\infty} \frac{2x+1}{\sqrt[3]{x^3-2}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt[3]{1-\frac{2}{x^3}}} = 2$

22. Find $\lim_{x\to-\infty} \frac{3x+2}{\sqrt{x^2-1}} = \lim_{x\to-\infty} \frac{3+\frac{2}{x}}{-\sqrt{1-\frac{1}{x^2}}} = -3$

23. Evaluate: $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

    Notice that, for $x > 0$, $\frac{1}{x} - \frac{1}{|x|} = \frac{1}{x} - \frac{1}{x} = 0$. Therefore, $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = 0$

    Side Remark: if $x < 0$, then $\frac{1}{x} - \frac{1}{|x|} = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$

24. Let

$$h(x) = \begin{cases} x, & \text{if } x < 0 \\ x^2, & \text{if } 0 < x \le 2 \\ 8-x, & \text{if } x > 2 \end{cases}$$

    Evaluate the following, if they exist.

    (a) $\lim_{x\to 0^+} h(x) = 0$ (Use $h(x) = x^2$)
    (b) $\lim_{x\to 0^-} h(x) = 0$ (Use $h(x) = x$)
    (c) $\lim_{x\to 1} h(x) = 1$ (Use $h(x) = x^2$)
    (d) $\lim_{x\to 2^+} h(x) = 6$ (Use $h(x) = 8-x$)
    (e) $\lim_{x\to 2^-} h(x) = 4$ (Use $h(x) = x^2$)
    (f) $\lim_{x\to 2} h(x)$ By (d) and (e), DNE

25. Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}$

26. Show that $\lim_{x\to 0} \sqrt{x}\, \mathrm{e}^{\sin(\pi/x)} = 0$

First, note that: $\sin(\pi/x)$ is always between 1 and -1, for all $x \neq 0$. This means that

$$\sqrt{x}\mathrm{e}^{-1} \leq \sqrt{x}\, \mathrm{e}^{\sin(\pi/x)} \leq \sqrt{x} \cdot \mathrm{e}^{1}$$

and the first and last expressions go to zero as x goes to zero. By the Squeeze Theorem, $\lim_{x\to 0} \sqrt{x}\, \mathrm{e}^{\sin(\pi/x)} = 0$.

27. Evaluate $\lim_{x\to -\infty} \frac{r^4 - r^2 + 1}{r^5 + r^3 - r}$

Divide numerator and denominator by r^5, and we get:

$$\lim_{x\to -\infty} \frac{\frac{1}{r} - \frac{1}{r^3} + \frac{1}{r^5}}{1 + \frac{1}{r^2} - \frac{1}{r^4}} = 0$$

28. Evaluate $\lim_{x\to\infty} \frac{1-\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} - 1}{\frac{1}{\sqrt{x}} + 1} = -1$

29. Evaluate $\lim_{x\to -\infty} \frac{6t^2 + 5t}{(1-t)(2t-3)}$ It might be easiest to multiply the denominator out. Doing that, and dividing numerator and denominator by t^2 gives an answer of -3.

30. Evaluate $\lim_{x\to\infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3}$ Divide numerator and denominator by t^3 to get an answer of $\frac{7}{2}$