Exam 2 Review Solutions

1. True or False, and explain:

   (a) The derivative of a polynomial is a polynomial.

   True. A polynomial is a function of the form $a_0+a_1x+a_2x^2+\ldots a_nx^n$, and its derivative will also have integer powers of x by the Power Rule, nx^{n-1}.

   (b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$

   True:

$$\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}(f(x))^{1/2} = \frac{1}{2}(f(x))^{-1/2}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

   (c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.

   False. The notation, $\sec^{-1}(x)$ is for the inverse secant function, which is not the reciprocal of the secant.

   For extra practice, to get the formula for the derivative of $y = \sec^{-1}(x)$:

$$\sec(y) = x$$

   From this, draw a right triangle with one acute angle labelled y, the hypotenuse x and the adjacent length x. This gives the length of the side opposite: $\sqrt{x^2-1}$. Now differentiate:

$$\sec(y)\tan(y)\frac{dy}{dx} = 1$$

   From the triangle, $\sec(y) = x$ and $\tan(y) = \sqrt{x^2-1}$, so:

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

   (d) $\frac{d}{dx}(10^x) = x10^{x-1}$

   False. The Power Rule can only be used for x^n, not a^x. The derivative is $10^x\ln(10)$.

   (e) If $y = \ln|x|$, then $y' = \frac{1}{x}$.

   TRUE. To see this, re-write the function:

$$y = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad y' = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}\cdot(-1) & \text{if } x < 0 \end{cases}$$

   From which we see that $y' = \frac{1}{x}$, $x \neq 0$.

   (f) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is:

$$y - 1 = 2x(x-1)$$

False. This is the equation of a parabola, not a line. The derivative, $y' = 2x$ gives a formula for the slope of the tangent line, and is not the slope itself. To get the slope, we evaluate the derivative at $x = 1$, which gives $y' = 2$. The slope of the tangent line is therefore $y - 1 = 2(x - 1)$.

(g) If $y = e^2$, then $y' = 2e$

False. e^2 is a constant, so the derivative is zero.

(h) If $y = |x^2 + x|$, then $y' = |2x + 1|$.

False. First, rewrite y, then differentiate:

$$y = \begin{cases} x^2 + x & \text{if } x \le 0 \text{ or } x \ge 1 \\ -(x^2 + x) & \text{if } 0 < x < 1 \end{cases} \quad \Rightarrow \quad y' = \begin{cases} 2x + 1 & \text{if } x < 0 \text{ or } x > 1 \\ -(2x + 1) & \text{if } 0 < x < 1 \end{cases}$$

Compare this to $|2x + 1|$:

$$|2x + 1| = \begin{cases} 2x + 1 & \text{if } x \ge -1/2 \\ -(2x + 1) & \text{if } x < -1/2 \end{cases}$$

By the way, we also note that y is NOT differentiable at $x = 0$ or at $x = 1$ by checking to see what the derivatives are approaching as $x \to 0$ and as $x \to 1$.

(i) If $y = ax + b$, then $\frac{dy}{da} = x$

True. $\frac{dy}{da}$ means that we treat a as an independent variable, and x, b as constants.

2. Find the equation of the tangent line to $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.

We need to find the slope, $\left.\frac{dy}{dx}\right|_{x=3/2, y=3/2}$

$$3x^2 + 3y^2 y' = 3y + 3xy' \Rightarrow y'(3y^2 - 3x) = 3y - 3x^2 \Rightarrow y' = \frac{y - x^2}{y^2 - x}$$

Substituting $x = 3/2, y = 3/2$ gives $y' = -1$, so the equation of the tangent line is $y - 3/2 = -1(x - 3/2)$

3. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

A cute chain rule problem! Here we go- the derivative is:

$$f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

Now subsitute $x = 0$ and evaluate:

$$f'(f(f(f(0)))) \cdot f'(f(f(0))) \cdot f'(f(0)) \cdot f'(0)$$

$$f'(0) \cdot f'(0) \cdot f'(0) \cdot f'(0) = 2^4 = 16$$

4. If $f(x) = 2x + \mathrm{e}^x$, find the equation of the tangent line to the INVERSE of f at $(1, 0)$.

   First, we verify that $(0, 1)$ is on the graph of f:

   $$f(0) = 2 \cdot 0 + \mathrm{e}^0 = 1$$

   We know that, if $f'(0) = m$, then $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{m}$.

   Now, $f'(x) = 2 + \mathrm{e}^x$, so $f'(0) = 3$. Therefore, the slope of the tangent line to the inverse of f at $x = 1$ is $\frac{1}{3}$, and the equation is then:

   $$y - 0 = \frac{1}{3}(x - 1) \text{ or } y = \frac{1}{3}x - \frac{1}{3}$$

5. Derive the formula for the derivative of $y = \cos^{-1}(x)$ using implicit differentiation.

   First, $\cos(y) = x$, so that implicit differentiation gives $-\sin(y)y' = 1$, so $y' = -1/\sin(y)$. Now to convert this back to x, draw a right triangle with y as one of the acute angles. Label the adjacent side as x, hypotenuse as 1, so the length of the side opposite is $\sqrt{1 - x^2}$. This gives:

   $$y' = \frac{-1}{\sqrt{1 - x^2}}$$

6. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

   Implicit differentiation gives:

   $$\frac{1}{2}y^{-1/2}y' + y^2 + 2xyy' = 0$$

   Now we could solve for y' now, or substitute $x = 4, y = 1$:

   $$\frac{1}{2}1^{-1/2}y' + 1^2 + 2(4)(1)y' = 0 \Rightarrow y' = -2/17$$

   The equation of the tangent line is $y - 1 = \frac{-2}{17}(x - 4)$

7. If $s^2t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

   The notation $\frac{dt}{ds}$ means that we are treating t as a function of s. Therefore, we have:

   $$2st + s^2\frac{dt}{ds} + 3t^2\frac{dt}{ds} = 0 \Rightarrow \frac{dt}{ds} = \frac{-2st}{s^2 + 3t^2}$$

   For the second part, we have two choices. One choice is to treat s as a function of t and differentiate:

   $$2s\frac{ds}{dt}t + s^2 + 3t^2 = 0 \Rightarrow \frac{ds}{dt} = \frac{-(s^2 + 3t^2)}{2st}$$

Another method is to realize that:

$$\frac{ds}{dt} = \frac{1}{\frac{dt}{ds}} = \frac{1}{\frac{-2st}{s^2+3t^2}} = -\frac{s^2+3t^2}{2st}$$

Cool!

8. If $y = x^3 - 2$ and $x = 3z^2 + 5$, then find $\frac{dy}{dz}$.

   We see that $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz}$, so we calculate $\frac{dy}{dx}$ and $\frac{dx}{dz}$:

   $$\frac{dy}{dx} = 3x^2, \quad \frac{dx}{dz} = 6z$$

   so that

   $$\frac{dy}{dz} = 3x^2 \cdot 6z = 3(3z^2+5)^2 \cdot 6z = 18z(3z^2+5)^2$$

9. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$?

   The unknown in the problem is a point on the parabola $y = x^2$. Let's label that point as (a, a^2). Now our goal is to find a.

   First, the line will go through both (a, a^2) and $(4, 15)$, so the slope will satisfy:

   $$m = \frac{a^2 - 15}{a - 4}$$

   Secondly, the line will be a tangent line, so the slope will also be $m = 2a$. Equating these, we can solve for a:

   $$\frac{a^2-15}{a-4} = 2a \Rightarrow a^2 - 15 = 2a(a-4) \Rightarrow a^2 - 8a + 15 = 0 \Rightarrow a = 3, a = 5$$

   Since we're moving from left to right, we would choose the smaller of these, $a = 3$.

10. A particle moves in the plane according to the law $x = t^2 + 2t$, $y = 2t^3 - 6t$. Find the slope of the tangent line when $t = 0$.

    The slope is $\frac{dy}{dx}$, but we can only compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Note however, that

    $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 6}{2t + 2}$$

    So, at $t = 0$, $\frac{dy}{dx} = -3$.

11. Find the coordinates of the point on the curve $y = (x-2)^2$ at which the tangent line is perpendicular to the line $2x - y + 2 = 0$.

    First, recall that two slopes are perpendicular if they are negative reciprocals (like $-3, \frac{1}{3}$).

    The slope of the given line is 2, so we want a slope of $-\frac{1}{2}$.

    The x that will provide this slope is found by differentiating:

    $$y' = 2(x-2) \Rightarrow 2(x-2) = -\frac{1}{2} \Rightarrow x = \frac{7}{4} \text{ from which } y = \frac{1}{16}$$

12. For what value(s) of A, B, C does the polynomial $y = Ax^2 + Bx + C$ satisfy the differential equation:

    $$y'' + y' - 2y = x^2$$

    Hint: If $ax^2 + bx + c = 0$ for ALL x, then $a = 0, b = 0, c = 0$.

    As we did in class, compute the derivatives of y and substitute into the equation:

    $$y' = 2Ax + B, \quad y'' = 2A$$

    so that:

    $$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

    Now collect coefficients to get:

    $$(-1 - 2A)x^2 + (2A - 2B)x + (2A - 2C) = 0 \text{ for all } x$$

    so $(-1-2A) = 0, (2A-2B) = 0, (2A-2C) = 0$. This gives the solution, $A = -\frac{1}{2}$, $B = -\frac{1}{2}, C = -\frac{1}{2}$.

13. If $V = \sin(w)$, $w = \sqrt{u}$, $u = t^2 + 3t$, compute: The rate of change of V with respect to w, the rate of change of V with respect to u, and the rate of change of V with respect to t.

    - $\dfrac{dV}{dw} = \cos(w)$
    - $\dfrac{dV}{du} = \dfrac{dV}{dw} \cdot \dfrac{dw}{du} = \cos(w) \cdot \dfrac{1}{2\sqrt{u}} = \dfrac{\cos(\sqrt{u})}{2\sqrt{u}}$
    - $\dfrac{dV}{dt} = \dfrac{dV}{dw} \cdot \dfrac{dw}{du} \cdot \dfrac{du}{dt} = \cos(w) \cdot \dfrac{1}{2\sqrt{u}} \cdot (2t+3) = \dfrac{\cos(\sqrt{t^2+3t})}{2\sqrt{t^2+3t}} \cdot (2t+3)$

14. Find all value(s) of k so that $y = e^{kt}$ satisfies the differential equation: $y'' - y' - 2y = 0$.

    First, differentiate y, then substitute:

    $$y = e^{kt} \quad \Rightarrow \quad y' = ke^{kt} \quad \Rightarrow \quad y'' = k^2e^{kt}$$

so that:

$$k^2 e^{kt} - ke^{kt} - 2e^{kt} = 0 \Rightarrow e^{kt}(k^2 - k - 2) = 0$$

Since $e^{kt} = 0$ has no solution, the only solution(s) come from:

$$k^2 - k - 2 = 0 \quad \Rightarrow \quad (k+1)(k-2) = 0$$

so $k = -1$, $k = 2$ are the two values of k.

15. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

    Implicit differentiation gives:

$$2x + 4yy' = 0 \Rightarrow y' = \frac{x}{2y}$$

    To have a slope of 1 will mean that: $\frac{-x}{2y} = 1$, so that $x = -2y$. Substituting this back into the equation of the ellipse, we get:

$$(2y)^2 + 2y^2 = 1 \Rightarrow 6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}}, \text{ so that } x = \pm\frac{-2}{\sqrt{6}}$$

    The coordinates are either: $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ or $(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$

16. Differentiate. You may assume that y is a function of x, if not already defined explicitly.

    (a) $y = \log_3(\sqrt{x} + 1)$ Use the Chain Rule:

$$y' = \frac{1}{(\sqrt{x}+1)\ln(3)} \cdot \frac{1}{2\sqrt{x}}$$

    (b) $\sqrt{2xy} + xy^3 = 5$ (and solve for $\frac{dy}{dx}$)
    Before solving for y', we get:

$$\frac{1}{2}(2xy)^{-1/2}(2y + 2xy') + y^3 + 3xy^2 y' = 0$$

$$y' = \frac{-(y(2xy)^{-1/2} + y^3)}{x(2xy)^{-1/2} + 3xy^2}$$

    (c) $y = \sqrt{x^2 + \sin(x)}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2 + \sin(x)\right)^{-1/2}(2x + \cos(x))$$

    (d) $y = e^{\cos(x)} + \sin(5^x)$

$$y' = e^{\cos(x)}(-\sin(x)) + \cos(5^x) \cdot 5^x \ln(5)$$

(e) $y = \cot(3x^2 + 5)$

$$y' = -\csc^2(3x^2+5)(6x) = -6x\csc^2(3x^2+5)$$

(f) $y = x^{\cos(x)}$
Use logarithmic differentiation: $\ln(y) = \cos(x) \cdot \ln(x)$, so that

$$\frac{1}{y}y' = -\sin(x)\ln(x) + \cos(x) \cdot \frac{1}{x}$$

Multiply both sides of the equation by y, and back substitute $y = x^{\cos(x)}$ to get:

$$y' = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$$

(g) $y = \sqrt{\sin(\sqrt{x})}$

$$y' = \frac{1}{2}(\sin(x^{1/2}))^{-1/2}\cos(x^{1/2})\frac{1}{2}x^{-1/2}$$

(h) $\sqrt{x} + \sqrt[3]{y} = 1$

$$\frac{1}{2}x^{-1/2} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y' = -\frac{3y^{2/3}}{2x^{1/2}}$$

(i) $x\tan(y) = y - 1$

$$\tan(y) + x\sec^2(y)y' = y'$$

$$\frac{\tan(y)}{1 - x\sec^2(y)} = y'$$

(j) $y = \sqrt{x}\,e^{x^2}(x^2+1)^{10}$ (Hint: Logarithmic Diff)
First, we rewrite so that:

$$\ln(y) = \ln(\sqrt{x}\,e^{x^2}(x^2+1)^{10})$$

Use the rules of logarithms to re-write this as the sum:

$$\ln(y) = \frac{1}{2}\ln(x) + x^2\ln(e) + 10\ln(x^2+1) = \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1)$$

So far, we've only done algebra. Now it's time to differentiate:

$$\frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\cdot\frac{1}{x^2+1}\cdot 2x$$

Simplifying, multiplying through by y:

$$y' = y\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

Finally, back substitute y:

$$y' = \sqrt{x}\,e^{x^2}(x^2+1)^{10} \cdot \left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

(k) $y = \sin^{-1}\left(\tan^{-1}(x)\right)$
This is a composition, so use the chain rule:

$$y' = \frac{1}{\sqrt{1-\left(\tan^{-1}(x)\right)^2}} \cdot \frac{1}{x^2+1}$$

(l) $y = \ln|\csc(3x) + \cot(3x)|$
Recall that the derivative of $\ln|x|$ is $\frac{1}{x}$, so using the Chain Rule:

$$y' = \frac{1}{\csc(3x)+\cot(3x)} \cdot \left[-\csc(3x)\cot(3x)\cdot 3 - \csc^2(3x)\cdot 3\right]$$

Which can be simplified:

$$y' = \frac{-3\csc(3x)(\cot(3x)+\csc(3x))}{\csc(3x)+\cot(3x)} = -3\csc(3x)$$

(m) $y = \frac{-2}{\sqrt[4]{t^3}}$ First, note that $y = -2t^{-3/4}$ so $y' = \frac{3}{2}t^{-7/4}$

(n) $y = x3^{-1/x}$

$$y' = 3^{-1/x} + x3^{-1/x}\ln(3)\cdot x^{-2}$$

(o) $y = x\tan^{-1}(\sqrt{x})$
Overall, use the product rule (then a chain rule):

$$y' = \tan^{-1}(\sqrt{x}) + x\cdot\left(\frac{1}{(\sqrt{x})^2+1}\cdot\frac{1}{2\sqrt{x}}\right) == \tan^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2(x^2+1)}$$

(p) $y = e^{2^{e^x}}$ Before putting in the values, note that this derivative will be in the form:

$$y' = e^{(\,)} \cdot \frac{d}{dx}(2^{(\,)}) \cdot \frac{d}{dx}e^x = e^{(\,)} \cdot 2^{(\,)}\ln(2)\cdot e^x$$

Putting in the appropriate expressions gives us:

$$y' = e^{2^{e^x}}\,2^{e^x}\ln(2)e^x$$

(q) Let a be a positive constant. $y = x^a + a^x$

$$y' = ax^{a-1} + a^x \ln(a)$$

(r) $x^y = y^x$

We must use logs first, since the exponents have x and y:

$$\ln(x^y) = \ln(y^x) \Rightarrow y\ln(x) = x\ln(y) \Rightarrow y'\ln(x) + y \cdot \frac{1}{x} = \ln(y) + x \cdot \frac{1}{y} y'$$

Now isolate and solve for y':

$$y'\left(\ln(x) - \frac{x}{y}\right) = \ln(y) - \frac{y}{x} \Rightarrow y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}} = \frac{y(x\ln(y) - y)}{x(y\ln(x) - x)}$$

(s) Rewrite first: $y = \ln\left(\sqrt{\frac{3x+2}{3x-2}}\right) = \frac{1}{2}\left(\ln(3x+2) - \ln(3x-2)\right)$ Now y' can be computed:

$$y' = \frac{1}{2}\left(\frac{3}{3x+2} - \frac{3}{3x-2}\right) = \frac{1}{2}\left(\frac{3(3x-2) - 3(3x+2)}{(3x+2)(3x-2)}\right) = \frac{-6}{(3x+2)(3x-2)}$$

(t)

$$f'(x) = 2xg(2x+5) + x^2 g'(2x+5)(2) = 2x\,g(2x+5) + 2x^2\,g'(2x+5)$$

$$f''(x) = 2\,g(2x+5) + 2x\,g'(2x+5)(2) + 4x\,g'(2x+5) + 2x^2\,g''(2x+5)(2)$$

Simplify to: $f''(x) = 2\,g(2x+5) + 8x\,g'(2x+5) + 4x^2\,g''(2x+5)$

17. (This was left over from Section 2.9)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1+2(x+h)^2} - \sqrt{1+2x^2}}{h}$$

Multiply by the conjugate, simplify, take the limit:

$$\lim_{h\to 0} \frac{\sqrt{1+2(x+h)^2} - \sqrt{1+2x^2}}{h} \frac{\sqrt{1+2(x+h)^2} + \sqrt{1+2x^2}}{\sqrt{1+2(x+h)^2} + \sqrt{1+2x^2}} =$$

$$\lim_{h\to 0} \frac{1 + 2(x+h)^2 - (1+2x^2)}{h(\sqrt{1+2(x+h)^2} + \sqrt{1+2x^2})} = \lim_{h\to 0} \frac{h(4x+2h)}{h(\sqrt{1+2(x+h)^2} + \sqrt{1+2x^2})} =$$

$$\frac{4x}{2\sqrt{1+2x^2}} = \frac{2x}{\sqrt{1+2x^2}}$$

We can now double-check our work by differentiating directly:

$$f'(x) = \frac{1}{2}(1+2x^2)^{-1/2}(4x) = \frac{2x}{\sqrt{1+2x^2}}$$

18. Fill in the blanks for the proof of *The Product Rule:*

$$
\begin{aligned}
\tfrac{d}{dx}\left(f(x)g(x)\right) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{g(x+h)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))}{h}\\
&= \lim_{h\to 0} g(x+h)\cdot\frac{f(x+h)-f(x)}{h}+f(x)\cdot\frac{g(x+h)-g(x)}{h}\\
&= g(x)f'(x)+f(x)g'(x)
\end{aligned}
$$