Practice with Riemann Sums

Overview

The Riemann sum approximates the area under the curve $y = f(x)$ (where $f(x) \geq 0$ for positive area), on the interval $[a.b]$ by subdividing the interval into n equal subintervals. The $x-$ values for the points we get are:

$$x_0 = a, \quad x_1, x_2, x_2, \ldots, x_n = b$$

where $\Delta x = \frac{b-a}{n}$ is th width of each interval. On each interval, we estimate the area by building a rectangle with Δx as the base, and $f(x_i^*)$ as the height. Then the sum of the areas of the rectangles is given by

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If we choose left or right endpoints, then we can be more explicit with the formula:

$$L_n = \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)\frac{b-a}{n} \qquad R_n = \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$$

Finally, recall that the definite integral is **defined** as the limit of a Riemann sum:

Sample practice problems:

1. For each of the following integrals, approximate the area under the curve using a Riemann sum with either left or right endpoints.

   Worked Example: For $f(x) = 1/x$ on the interval $[1, 6]$ using left endpoints.

   SOLUTION: $\Delta x = \frac{6-1}{n} = \frac{5}{n}$, so

$$L_n = \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)\frac{b-a}{n} = \sum_{i=1}^{n} \frac{1}{1 + (i-1)\frac{5}{n}}\frac{5}{n}$$

   Here are some for you to work out:

   (a) $f(x) = x^2$ on $[0, 2]$ using right endpoints.
   (b) $f(x) = \sqrt{x}$ on $[1, 5]$ using left endpoints.
   (c) $f(x) = e^x$ on $[0, 3]$ using right endpoints.

2. For each of the following Riemann sums, write an appropriate definite integral.

Worked Example: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \sin\left(\frac{6i}{n}\right) = \int_0^6 \sin(x)\,dx$

Here are some for you to work out:

(a) $\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(2+\frac{3i}{n}\right)^2 + 2\left(2+\frac{3i}{n}\right)\right]$ (Find two different integrals for this one!)

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \ln\left(1+\frac{3i}{n}\right)$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(3+\frac{2(i-1)}{n}\right)^3$