Review Sheet 1 Solutions

1. State the definition of $\int_a^b f(x)\,dx$ assuming equal subintervals and right endpoints.

   SOLUTION:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

   KEY IDEA: If the definition is asked for, you must use the Riemann sum, and NOT the Fundamental Theorem of Calculus.

2. True or False, and give a short reason:

   (a) The Alternating Series Test is sufficient to show that a series is conditionally convergent.
   FALSE, because if you go directly to the Alternating Series Test, you do not know if the series converges absolutely.

   (b) You can use the Integral Test to show that a series is absolutely convergent.
   TRUE. The Direct Comparison, Limit Comparison, Ratio Test and Integral tests were all used on positive series.

   (c) Consider $\sum a_n$. If $\lim_{n\to\infty} a_n = 0$, then the sum is said to converge.
   FALSE. For example, $1/n \to 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
   Although, if the terms do NOT go to zero, then the sum diverges by the Test for Divergence.

   (d) The sequence $a_n = 0.1^n$ converges to $\frac{1}{1-0.1}$
   FALSE. This is a sequence, not a series. The sequence converges to zero.

   (e) All continuous functions have derivatives.
   FALSE: For example, $f(x) = |x|$ is continuous, but is not differentiable at $x = 0$.
   Side remark: This problem was given to you to compare it against the next problem.

   (f) All continuous functions have antiderivatives.
   TRUE. That is the Fundamental Theorem of Calculus, Part I.

3. Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the $y-$axis.

   For the volume, we might use shells. In that case, the height of a shell is $\sqrt{x-1}$ and the radius is x. The volume is then:

$$\int_1^5 2\pi x\sqrt{x-1}\,dx$$

4. Write the area under $y = \sqrt[3]{1+x}$, $1 \le x \le 4$ as the limit of a Riemann sum (use right endpoints)
   SOLUTION: The i^{th} right endpoint is: $1 + \frac{3i}{n}$. The height at this endpoint is given by:

$$f\left(1 + \frac{3i}{n}\right) = \sqrt[3]{1 + \left(1 + \frac{3i}{n}\right)} = \sqrt[3]{2 + \frac{3i}{n}}$$

   Therefore, the integral as a Riemann sum is:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{2 + \frac{3i}{n}} \cdot \frac{3}{n}$$

Secondly, for the arc length, we need to differentiate y:

$$y' = \frac{1}{3}(1+x)^{-2/3} \quad \Rightarrow \quad (y')^2 = \frac{1}{9}(1+x)^{-4/3}$$

so the integral is:

$$\int_1^4 \sqrt{1 + \frac{1}{9(1+x)^{4/3}}}\, dx$$

5. Find the Taylor series for $f(x) = \sqrt{x}$ centered at $a = 9$.

SOLUTION: Look at the table to see if a pattern is showing up. This pattern is a little tricky, so the question would typically ask for a few terms of the sum (but it's good practice for pattern finding).

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{1/2}$	3
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2\cdot 3}$
2	$-\frac{1}{2^2}x^{-3/2}$	$-\frac{1}{2^2 3^3}$
3	$\frac{1\cdot 3}{2^3}x^{-5/2}$	$\frac{1\cdot 3}{2^3 3^5}$
4	$-\frac{1\cdot 3\cdot 5}{2^4}x^{-7/2}$	$-\frac{1\cdot 3\cdot 5}{2^4 3^7}$
5	$\frac{1\cdot 3\cdot 5\cdot 7}{2^5}x^{-9/2}$	$\frac{1\cdot 3\cdot 5\cdot 7}{2^5 3^9}$
$\vdots$		$\vdots$
n		$(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n 3^{2n-1}}$

with the pattern starting at $n = 2$. If you found the pattern, write the general formula. At a minimum, write down the first few terms as shown.

$$\sqrt{x} = 3 + \frac{1}{2\cdot 3}(x-9) - \frac{1}{2^2 3^3 \cdot 2}(x-9)^2 + \frac{1}{2^3 3^3 \cdot 3!}(x-9)^3 + \cdots$$

In general:

$$\sqrt{x} = 3 + \frac{1}{6}(x-9) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2^n 3^{2n-1} n!}(x-9)^n$$

6. Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} cos(t^2)\, dt$

SOLUTION: The general formula we got using the Fundamental Theorem of Calculus, Part I was:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Therefore, in this particular case,

$$\cos(25x^2)\cdot 5 + \cos\left(\cos^2(x)\right)\cdot \sin(x)$$

7. Let $f(x) = e^x$ on the interval $[0, 2]$. (a) Find the average value of f. (b) Find c such that $f_{\text{avg}} = f(c)$.

SOLUTION: Remember the theorem: If f is continuous on $[a, b]$, then there is a c in $[a, b]$ so that:

$$f_{avg} = f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

So, we first compute the average, then we'll find the c:

$$\frac{1}{2}\int_0^2 \mathrm{e}^x\,dx = \frac{\mathrm{e}^2-1}{2}$$

so that $c = \ln\left(\frac{\mathrm{e}^2-1}{2}\right) \approx 1.16$

8. Use a template series to find the series for $\int \cos(x^2)\,dx$.

   SOLUTION: The template series is the series for $\cos(x)$. In that formula, substitute x^2 for x, then integrate:

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!} \quad\Rightarrow\quad \int \cos(x^2)\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!} + C$$

9. Does the series converge (absolute or conditional), or diverge?

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n/2)}{n^2+4n}$ Note that $\cos(n/2) < 1$ for all n. Therefore, by direct comparison, $\left|\frac{\cos(n/2)}{n^2+4n}\right| < \frac{1}{n^2}$ Thus, the series converges absolutely by the direct comparison test.

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ Using the ratio test, $\displaystyle\lim_{n\to\infty}\frac{(n+1)^2+1}{5^{n+1}}\cdot\frac{5^n}{n^2+1} = \lim_{n\to\infty}\frac{(n+1)^2+1}{n^2+1}\cdot\frac{1}{5} = \frac{1}{5}$ The series converges absolutely by the ratio test.

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ By the ratio test, $\displaystyle\lim_{n\to\infty}\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\cdot\frac{3}{n+1} = 0$

10. Find the interval of convergence:

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{10^n}$ (Ratio) $\displaystyle\lim_{n\to\infty}\frac{(n+1)^2|x|^{n+1}}{10^{n+1}}\cdot\frac{10^n}{n^2|x|^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\cdot\frac{|x|}{10} = \frac{|x|}{10}$ The radius of convergence is 10. Check the endpoints:

    If $x = 10$, the sum is $\sum n^2$, which diverges. If $x = -10$, the sum is $\sum(-1)^n n^2$, which still diverges. The interval of convergence is therefore: $(-10, 10)$

    (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n3^n}$ (Ratio) $\displaystyle\lim_{n\to\infty}\frac{|3x-2|^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{|3x-2|^n} = \lim_{n\to\infty}\frac{n}{n+1}\frac{|3x-2|}{3} = \frac{|3x-2|}{3}$

    Side Remark: If you were asked for the radius of convergence, you would have one more step- factor out the 3 in the numerator so that

$$\frac{3|x-2/3|}{3} < 1 \quad\Rightarrow\quad \left|x-\frac{2}{3}\right| < 1$$

    so the radius of convergence is 1 (not 3). That doesn't affect the endpoint computations, however. The interval so far is: $(-1/3, 5/3)$, so now check the endpoints.

    If $x = -\frac{1}{3}$, the sum becomes: $\sum\frac{(-1)^n}{n}$, which converges (conditionally). If $x = \frac{5}{3}$, the sum becomes the Harmonic Series, which diverges.

    The interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

Be careful with the indices on this one. If the n^{th} term has index $2n-1$, then the $n+1^{\text{st}}$ index is $2(n+1)-1 = 2n+1$. Furthermore,

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1\cdot 2\cdot 3\cdots(2n-2)(2n-1)}{1\cdot 2\cdot 3\cdots(2n-2)(2n-1)(2n)(2n+1)} = \frac{1}{(2n)(2n+1)}$$

Now apply the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{|x|^{2n-1}} = \lim_{n\to\infty} \frac{x^2}{(2n)(2n+1)} = 0$$

for all x. Therefore, the interval of convergence is the set of all real numbers.

11. Evaluate the integral.

(a) $\displaystyle\int \frac{1}{y^2-4y-12}\,dy$

SOLUTION: Use partial fractions, where

$$\frac{1}{y^2-4y-12} = \frac{1}{(y+2)(y-6)} = \frac{A}{y+2} + \frac{B}{y-6}$$

Then:

$$1 = A(y-6) + B(y+2) \qquad \Rightarrow \quad A = -1/8, \quad B = -1/8$$

Now integrate: $\displaystyle\int \frac{1}{y^2-4y-12}\,dy = -\frac{1}{8}\ln|y+2| + \frac{1}{8}\ln|y-6|$

(b) $\displaystyle\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\,dx$

The first integral is a straight u, du substitution with $u = 3x+1$. The second integralcan be split into two:

$$\int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

On the first integral, use u, du substitution with $u = x^2+9$, and on the second integral, use either trig substitution (with triangles), or use the page of formulas. For extra practice, the trig substitution is also given below.

Use the triangle where $\tan(\theta) = x/3$, so the hypotenuse is $\sqrt{x^2+9}$, $\sec(\theta) = \frac{\sqrt{x^2-9}}{3}$, and $dx = 3\sec^2(\theta)$. The integral becomes:

$$\int \frac{3}{x^2+9}\,dx = \int \frac{9\sec^2(\theta)}{9\sec^2(\theta)}\,d\theta = \theta = \tan^{-1}(x/3)$$

Altogether, we get:

$$\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\,dx = \frac{2}{3}\ln|3x+1| + \ln(x^2+9) + \tan^{-1}(x/3) + C$$

(c) $\displaystyle\int x^2\cos(3x)\,dx$ Use integration by parts using a table:

$+$	x^2	$\cos(3x)$
$-$	$2x$	$\frac{1}{3}\sin(3x)$
$+$	2	$-\frac{1}{9}\cos(3x)$
$-$	0	$-\frac{1}{27}\sin(3x)$

$$\Rightarrow \int x^2\cos(3x)\,dx = \frac{1}{3}x^2\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$$

(d) $\int_{-2}^{2} |x-1|\,dx$ Here, we want to split the integral at $x=1$:

$$\int_{-2}^{2} |x-1|\,dx = \int_{-2}^{1} -x+1\,dx + \int_{1}^{2} x-1\,dx = 5$$

(e) $\int \frac{dx}{x\ln(x)}$ Use a u, du substitution with $u=\ln(x)$, $du = \frac{1}{x}\,dx$. This gives:

$$\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}\,du = \ln|u| = \ln(\ln(x)) + C$$

(f) $\int x\sqrt{x-1}\,dx$

Let's use $u = x-1$ so that $x = u+1$ and $du = dx$ so that the integral becomes

$$\int (u+1)u^{1/2}\,du = \int u^{3/2} + u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

(g) $\int_{0}^{\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx$

SOLUTION: Before you integrate, consider what the integrand does as x gets large. In this case, the fraction will behave like $x^2/x^{3/2} = \sqrt{x}$, and that will diverge. Unfortunately, we can't do a direct comparison, but we can integrate and expect divergence. Let $u = 1+x^3$ so that $du = 3x^2\,dx$, and

$$\int \frac{x^2}{\sqrt{1+x^3}}\,dx = \frac{1}{3}\int u^{-1/2}\,du = \frac{2}{3}u^{1/2}$$

And taking the limit as $u \to \infty$ will be infinity, so the integral diverges.

(h) $\int \sin^2(x)\cos^3(x)\,dx$

SOLUTION: Reserve one of the cosines for $u = \sin(x)$, $du = \cos(x)\,dx$, and re-write the remaining $\cos^2(x) = 1 - \sin^2(x)$:

$$\int u^2(1-u^2)\,du = \int u^2 - u^4\,du = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$$

(i) $\int \sin^{-1}(x)\,dx$

SOLUTION: Use integration by parts with $\sin^{-1}(x)$ in the center column.

sign	u	dv
$+$	$\sin^{-1}(x)$	1
$-$	$1/\sqrt{1-x^2}$	x

$$\Rightarrow \quad x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

For this integral, use $u = 1-x^2$, $du = -2x\,dx$, and

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

(j) $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$

SOLUTION: This is a trig substitution problem, with $x = 3\sec(\theta)$ (so we can also get the right triangle with hypotenuse x and length of the side adjacent θ is 3). Then $dx = 3\sec(\theta)\tan(\theta)d\theta$, and

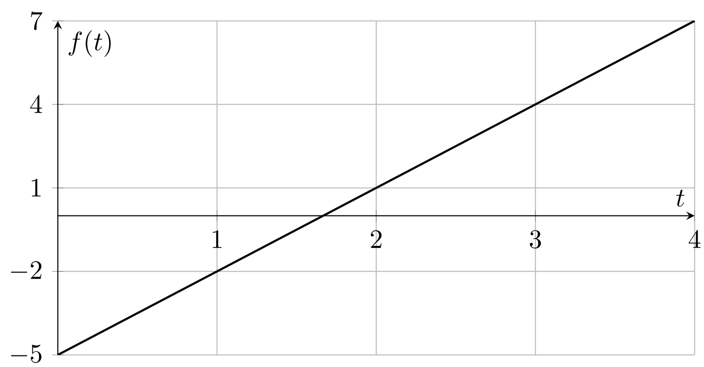
$$\int \frac{3\sec(\theta)\tan(\theta)d\theta}{\sqrt{9(\sec^2(\theta)-1)}} = \int \frac{3\sec(\theta)\tan(\theta)d\theta}{3\tan(\theta)} = \int \sec(\theta)\, d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

Using the right triangle described above, we back substitute to get x again:

$$\int \frac{dx}{\sqrt{x^2-9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

12. The velocity function is $v(t) = 3t-5$, $0 \le t \le 3$ (a) Find the displacement. (b) Find the distance traveled.

SOLUTION: Probably easiest to think of this geometrically.



The $x-$intercept is where $5t-3=0$, or $t=5/3$. Therefore, the areas of the two triangles shown are:

$$\frac{1}{2}\cdot 5\cdot\frac{5}{3} = \frac{25}{6} \qquad \text{and} \qquad \frac{1}{2}\cdot 4\cdot\left(3-\frac{5}{3}\right) = \frac{8}{3}$$

Therefore, the displacement between $0 \le t \le 3$ is: $-\dfrac{25}{6} + \dfrac{8}{3} = \dfrac{-25+16}{6} = -\dfrac{9}{6} = -\dfrac{3}{2}$

And the distance traveled: $\dfrac{25}{6} + \dfrac{8}{3} = \dfrac{25+16}{6} = \dfrac{41}{6}$

13. Set up the integral that will give the length of the graph of $y = \sqrt{x-x^2} + \sin^{-1}(\sqrt{x})$, with $0 \le x \le 1$.

We want to set up the integral for the arc length:

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Let's work on getting and simplifying the integrand first.

$$1 + (f'(x))^2 = 1 + \left[\frac{1}{2}(x-x^2)^{-1/2}(1-2x) + \frac{1}{\sqrt{1-(\sqrt{x}^2)}}\frac{1}{2}x^{-1/2}\right]^2$$

This simplifies a bit

$$1 + \frac{1}{4}\left[\frac{1-2x}{\sqrt{x-x^2}} + \frac{1}{\sqrt{x-x^2}}\right]^2 = 1 + \frac{4}{4}\frac{(1-x)^2}{x(1-x)} = 1 + \frac{1}{x} - 1 = \frac{1}{x}$$

And finishing, we were not asked to evaluate the integral:

$$L = \int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Side Note: Often on exams/quizzes, I will ask you to set up the integral only. This is because if an error is made in the setup, it can mess everything up past that point. Plus, this saves you some time on the exam.

14. Evaluate each sum. Hint on part (c): Partial Fractions

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

SOLUTION: Hopefully the sum looks familiar since we have even powers and even factorials-

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^{n+2}}{2^{2n}}$

SOLUTION: This is a geometric series in disguise.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^{n+2}}{2^{2n}} = \sum_{n=1}^{\infty} (-9)\left(-\frac{3}{4}\right)^n = \frac{27/4}{1+(3/4)} = \frac{27}{7}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

SOLUTION: Using the hint, we can expand the sum using partial fractions.

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \qquad \Rightarrow \qquad 1 = A(n+3) + Bn$$

Solving for A, B, we get:

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

Writing out the sum (factoring out the 1/3), we can write down S_n:

$$S_n = \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+3}$$

Focusing on this sum, terms are canceling out:

$$\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots +$$

$$\cdots + \left(\frac{1}{n-3} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

It looks like we're left with:

$$S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

As $n \to \infty$, $S_n \to \frac{11}{3 \cdot 6} = \frac{11}{18}$
This was an example of a telescoping series, which is good for practice.

15. From our definition,

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 1} = 4$$