Final Exam Review
Calculus II
Sheet 2

1. True or False, and give a short reason:

   (a) If f has a discontinuity at 0, then $\int_{-1}^{1} f(x)\,dx$ does not exist.
   FALSE. Back in Chapter 1, we simply said that the Fundamental Theorem of Calculus does not apply, but then in Section 3.7 (improper integrals), we learned how to define an integral at a point where the integrand is not continuous (with limits on the bounds).

   (b) The Ratio Test will not give a conclusive result for $\sum \frac{2n+3}{3n^4+2n^3+3n+5}$
   TRUE. The ratio test fails for $p-$like series (the limit will be 1). To show convergence, use a direct or limit comparison (Limit comparison with $1/n^3$)

   (c) If $\displaystyle\sum_{n=k}^{\infty} a_n$ converges for some large k, then so will $\displaystyle\sum_{n=1}^{\infty} a_n$.
   TRUE. The first few terms of a sum are irrelevant when looking at whether or not the sum converges (although they will effect what the sum converges to).

   (d) If f is continuous on $[0,\infty)$ and $\displaystyle\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.
   FALSE. For example, $1/(x-1)$. (The idea here is that functions must go to zero fast enough).

   (e) If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, then $\displaystyle\int_0^3 xf(x^2)\,dx = 4$.
   FALSE.

$$\int_0^3 xf(x^2)\,dx \Rightarrow \begin{array}{rl} u & = x^2 \\ (1/2)\,du & = dx \\ x=0 & \Rightarrow u=0 \\ x=3 & \Rightarrow u=9 \end{array} \Rightarrow \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2$$

2. Short Answer:

   (a) Suppose the series $\sum c_n 3^n$ converges. Will $\sum c_n(-2)^n$ also converge? For what values of x will the series $\sum c_n(x-2)^n$ converge?
   SOLUTION: For the first part of the question, we can look as if it were a power series $\sum c_n x^n$ that converged at $x=3$. Therefore, the series would converge for all $|x|<3$, and $x=-2$ is within that range. On the other hand, if we think of the series as $\sum c_n(x-2)^n$, then the series converges for all x so that $|x-2|<3$, or at least within the interval $(-1,5]$ (the convergence at $x-2=3$ might be conditional, that's why we did not include $x=-1$).

   (b) If $\sum a_n$, $\sum b_n$ are series with positive terms, and a_n, b_n both go to zero as $n\to\infty$, then what can we conclude if $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = 0$?
   SOLUTION: We can conclude that the terms of $\sum a_n$ are going to zero faster than b_n. Thus, if $\sum b_n$ is convergent, so is $\sum a_n$, and if $\sum a_n$ is divergent, so is $\sum b_n$.

   (c) What is the derivative of $\sin^{-1}(x)$? Of $\tan^{-1}(x)$? What is the antiderivative of each?
   SOLUTION: The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$
   To integrate either, use integration by parts. For $\sin^{-1}(x)$,

$$\begin{array}{ccc} + & \sin^{-1}(x) & 1 \\ - & 1/\sqrt{1-x^2} & x \end{array} \Rightarrow \int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

   For this integral, use $u = 1-x^2$, $du = -2x\,dx$ to get a final answer:

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

(d) Find the sum: $\sum_{n=1}^{\infty} \mathrm{e}^{-2n}$

SOLUTION: The sum of a geometric series, in its general form is:

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

In this case, $r = \mathrm{e}^{-2}$, so the sum is: $\frac{\mathrm{e}^{-2}}{1+\mathrm{e}^{-2}}$

3. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = 450\mathrm{e}^t$ bacteria per hour. How many bacteria will there be after three hours?

   SOLUTION: To find the net change, we integrate the rate of change over the given time interval, then add the initial population.

$$400 + \int_0^3 450\mathrm{e}^t \, dt = 400 + 450\mathrm{e}^t\Big|_0^3 = 400 + 450(e^3 - 1)$$

4. Suppose $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, and $h''(2) = 13$, and h'' is continuous. Evaluate $\int_1^2 h''(u)\, du$.

$$\int_1^2 h''(u)\, du = h'(2) - h'(1) = 5 - 2 = 3$$

5. Determine a definite integral representing: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$ [For extra practice, try writing the integral so that the right endpoint (or bottom bound) must be 5].

   SOLUTION: We need to find f so that

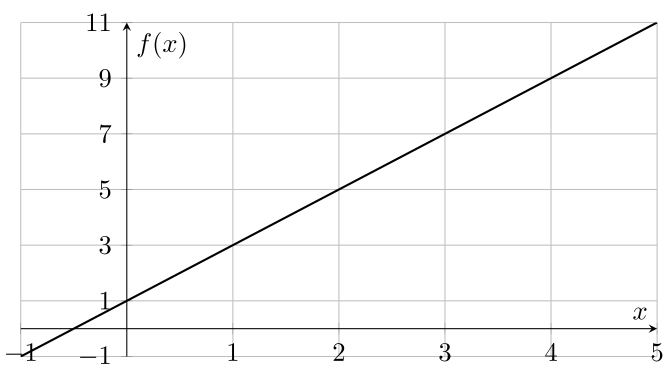
$$f\left(5 + \frac{3i}{n}\right) = \sqrt{1 + \frac{3i}{n}}$$

   Here is one: $f(x) = \sqrt{x-4}$. Our solution is:

$$\int_5^8 \sqrt{x-4}\, dx$$

6. Evaluate $\int_2^5 (1 + 2x)\, dx$ using geometry.

   Sketch the line, then find the resulting area:



If you recall the formula for the area of a trapezoid, you can use that to get that the area is 18. Otherwise, subtract the area of the large triangle from the area of the small triangle:

$$\frac{1}{2}\left(11 \cdot \frac{11}{2}\right) - \frac{1}{2}\left(7 \cdot \frac{7}{2}\right) = 18$$

7. For each function, find the Taylor series for $f(x)$ centered at the given value of a:

SOLUTION:

(a) $f(x) = 1 + x + x^2$ at $a = 2$ We need $f(2), f'(2), f''(2)$: $f(2) = 7$. $f'(x) = 1 + 2x$, so $f'(2) = 5$. $f''(x) = 2$ Now,

$$1 + x + x^2 = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 = 7 + 5(x-2) + (x-2)^2$$

(b) $f(x) = \frac{1}{x}$ at $a = 1$. We need to compute derivatives:

n	$f^n(x)$	$f^n(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-(3\cdot 2)x^{-4}$	$-(3\cdot 2)$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n n! x^{-(n+1)}$	$(-1)^n n!$

$$\Rightarrow \frac{f^{(n)}(1)}{n!} = (-1)^n \Rightarrow \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Alternatively, we could use the geometric series:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

8. Find a so that half the area under the curve $y = \frac{1}{x^2}$ lies in the interval $[1, a]$ and half of the area lies in the interval $[a, 4]$.

SOLUTION: We could set this up multiple ways- here is one way to do it:

$$\int_1^a \frac{1}{x^2}\, dx = \frac{1}{2} \int_1^4 \frac{1}{x^2}\, dx \Rightarrow -\frac{1}{a} + 1 = \frac{3}{8} \Rightarrow a = \frac{8}{5}$$

9. Compute the limit, by using the series for $\sin(x)$: $\lim_{x\to 0} \frac{\sin(x)}{x}$

SOLUTION: The series for the sine function is:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Therefore, the series for $\sin(x)/x$ is:

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots$$

To find the limit as $x \to 0$, we can evaluate the series at $x = 0$, which leaves the limit as 1.

10. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x$, $y = 4x - x^2$, about $x = 7$.

SOLUTION: First, find the region of interest. $y = 4x - x^2$ is an upside down parabola with $x-$intercepts at $x = 0, x = 4$. The point of intersection is $x = 4x - x^2 \Rightarrow 0 = 3x - x^2$, or $x = 0$ and $x = 3$. Now the region of interest is between $x = 0$, $x = 3$, above the line $y = x$ and below the parabola $y = 4x - x^2$. Rotate about $x = 7$, and we will use cylindrical shells (Washers would be possible, but messy!). The height of the cylinder is $(4x - x^2) - x = 3x - x^2$. The radius is $7 - x$. Therefore, the integral for the volume is:

$$\int_0^3 2\pi(7-x)(3x-x^2)\, dx$$

11. Evaluate each of the following:

[The purpose of this problem is to get you to see the differences in notation]

(a) $\frac{d}{dx}\int_{3x}^{\sin(x)} t^3\,dt$. By FTC, part I: $\sin^3(x)\cdot\cos(x) - (3x)^3\cdot 3$

(b) $\frac{d}{dx}\int_1^5 x^3\,dx = 0$ (this is the derivative of a constant)

(c) $\int_1^5 \frac{d}{dx}x^3\,dx = x^3\Big|_1^5 = 5^3 - 1 = 124$. This is FTC, part II.

12. Converge (absolute or conditional) or Diverge?

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{(n+1)(n+2)}$ This will behave like $\sum(-1)^n\frac{1}{n}$, which only converges conditionally.

We can use the limit comparison test (with $\frac{1}{n}$) to show that the series does not converge absolutely:

$$\lim_{n\to\infty}\frac{n}{(n+1)(n+2)}\cdot\frac{n}{1} = 1$$

The two series will diverge together, so the given series diverges.

Now we use the Alternating Series Test to show that it converges conditionally: Each term is clearly positive, for $n > 0$. Is it decreasing?

$$f(x) = \frac{x}{(x+1)(x+2)} \qquad f'(x) = \frac{2-x^2}{(x+1)^2(x+2)^2}$$

so the derivative is negative for $x > \sqrt{2}$ (or the terms of the series are decreasing for $n > 2$). Finally, show that the terms are going to zero:

$$\lim_{n\to}\frac{n}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{n}{n^2+3n+2} = \lim_{n\to\infty}\frac{1}{2n+3} = 0$$

(the last equality by l'Hospital's rule).

(b) $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

It looks like it should converge by comparing it to $\sum\frac{1}{n^2}$, so we'll try the limit comparison test:

$$\lim_{n\to\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot\frac{\sqrt{n^4}}{1} = \lim_{n\to\infty}\frac{\sqrt{n^6-n^4}}{n^3+2n^2+5}$$

(Don't use l'Hospital's rule!) Divide top and bottom by n^3:

$$\lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = 1$$

By the limit comparison test, the given series converges (absolutely, but that is irrelevant since the terms are all positive anyway).

(c) $\sum_{k=1}^{\infty}\frac{4^k+k}{k!}$ Use the ratio test:

$$\frac{4^{k+1}+(k+1)}{(k+1)!}\cdot\frac{k!}{4^k+k} = \frac{4^{k+1}+k+1}{(k+1)(4^k+k)} = \frac{4+\frac{k}{4^k}+\frac{1}{4^k}}{(k+1)(1+\frac{k}{4^k})}$$

The numerator approaches 4 as $k\to\infty$ and the denominator goes to ∞ as $k\to\infty$, so overall, the limit is 0. Therefore, this series converges (absolutely) by the Ratio Test.

13. Find the interval of convergence.

   (a) $\sum_{n=1}^{\infty} n^n x^n$ By the root test, $\lim_{n\to\infty} (n^n x^n)^{1/n} = \lim_{n\to\infty} nx = \infty$ Therefore, the only point of convergence is when $x = 0$. (The radius of convergence is also 0).

   Note: The root test is not used very often, but in this situation (where everything is raised to the n^{th} *power), this will make quick work of the problem.*

   (b) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

   Use the Ratio Test, as usual:

   $\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{|x|^n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{|x+2|}{4} = \frac{|x+2|}{4} < 1$ This means that the radius of convergence is 4, and the interval so far is $(-6, 2)$.

   Check the endpoints: If $x = 2$, then the sum is $\sum \frac{1}{n}$ which diverges. If $x = -6$, then the sum is $\sum \frac{(-1)^n}{n}$, which converges. The interval of convergence is therefor $-6 \le x < 2$.

   (c) $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$

   Use the Ratio Test:

   $\lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n|x-3|^n} = \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} \cdot 2|x-3| = 2|x-3| < 1$

   Therefore, the radius of convergence is $1/2$ and the interval is $5/2 < x < 7/2$. Now check endpoints:

   If $x = \frac{5}{2}$, the sum becomes $\sum \frac{(-1)^n}{\sqrt{n+3}}$, which converges by the Alternating Series test, and if $x = \frac{7}{2}$, the sum becomes $\sum \frac{1}{\sqrt{n+3}}$ which diverges (p-series).

14. Evaluate:

   (a) $\int_0^{\infty} \frac{1}{(x+2)(x+3)}\, dx$ By partial fractions,

   $$\int \frac{1}{(x+2)(x+3)}\, dx = \int \frac{1}{x+2} - \frac{1}{x+3}\, dx = \ln|x+2| - \ln|x+3| = \ln\left|\frac{x+2}{x+3}\right|$$

   As $x \to \infty$, $\ln\left|\frac{x+2}{x+3}\right| \to \ln(1) = 0$. Altogether we get:

   $$\int_0^{\infty} \frac{1}{(x+2)(x+3)}\, dx = 0 - \ln(2/3) = \ln(3/2)$$

   (b) $\int u(\sqrt{u} + \sqrt[3]{u})\, du$ Simplify algebraically first, to get $\int u^{3/2} + u^{4/3}\, du = \frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

   (c) $\int \frac{x^2}{(4-x^2)^{3/2}}\, dx$

   Use a triangle whose hypotenuse is 2, side opposite θ is x, and side adjacent is $\sqrt{4-x^2}$. Then, substitute $2\sin(\theta) = x$, $2\cos(\theta) = \sqrt{4-x^2}$, and we get:

   $$\int \frac{4\sin^2(\theta) \cdot 2\cos(\theta)}{2^3\cos^3(\theta)}\, d\theta = \int \tan^2(\theta)\, d\theta = \int \sec^2(\theta) - 1\, d\theta = \tan(\theta) - \theta$$

   Convert back using triangles to get: $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$

(d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$ Let $u = \tan^{-1}(x)$, so $du = \frac{1}{1+x^2}\,dx$. Then the integral becomes

$$\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1}(x))^2 + C$$

(e) $\displaystyle\int \frac{1}{\sqrt{x^2-4x}}\,dx$

"Complete the Square" in the denominator to get $x^2 - 4x = (x-2)^2 - 4$. Now, use a triangle whose hypotenuse is $x-2$, side adjacent is 2, and side opposite is $\sqrt{(x-2)^2-2^2}$. Then,

$$2\tan(\theta) = \sqrt{(x-2)^2-2^2}, \quad 2\sec(\theta) = x-2, \quad 2\sec(\theta)\tan(\theta)d\theta = dx$$

Substituting, we get:

$$\int \frac{1}{\sqrt{x^2-4x}}\,dx = \int \frac{2\sec(\theta)\tan(\theta)}{2\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

[NOTE: You'll be given the formulas as on the previous exam]. Final answer:

$$\ln\left|\frac{x-2}{2} + \frac{\sqrt{(x-2)^2-4}}{2}\right| + C$$

(f) $\displaystyle\int x^4\ln(x)\,dx$ Use integration by parts

$$\begin{array}{ccc} + & \ln(x) & x^4 \\ - & 1/x & (1/5)x^5 \end{array} \Rightarrow \frac{1}{5}x^5\ln(x) - \frac{1}{5}\int x^4\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$

(g) $\displaystyle\int e^{-x}\sin(2x)\,dx$. This is the type of integral for which we perform integration by parts twice to get the same integral on both sides of the equation:

$$\left|\begin{array}{c|c|c} + & \sin(2x) & e^{-x} \\ - & 2\cos(2x) & -e^{-x} \\ + & -4\sin(2x) & e^{-x} \end{array}\right| \Rightarrow \int e^{-x}\sin(2x)\,dx = -e^{-x}\sin(2x) - 2e^{-x}\cos(2x) - 4\int e^{-x}\sin(2x)\,dx$$

so that

$$\int e^{-x}\sin(2x)\,dx = -\frac{1}{5}e^{-x}\sin(2x) - \frac{2}{5}e^{-x}\cos(2x)$$

(h) $\displaystyle\int_0^3 \frac{1}{\sqrt{x}}\,dx$

Note that we have a vertical asymptote at $x = 0$, so

$$\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\int_T^3 x^{-1/2}\,dx = \lim_{T\to 0^+} 2x^{1/2}\Big|_T^3 = 2\sqrt{3} - 0 = 2\sqrt{3}$$

(i) $\displaystyle\int \sin^2 x\,dx$

We use the half angle formula. If you forget the formula, it can be quickly derived from the formula for $\cos(2x) = \cos^2(x) - \sin^2(x)$:

$$\cos(2x) = 1 - 2\sin^2(x) \quad\Rightarrow\quad -1 + \cos(2x) = -2\sin^2(x) \quad\Rightarrow\quad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Therefore,

$$\int \sin^2(x)\,dx = \frac{1}{2}\int 1 - \cos(2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

15. Find the surface area of the surface of revolution formed by rotating the graph of $y = x^2$ from $(1,1)$ to $(2,4)$ about the $y-$axis.

SOLUTION: Rotating about y instead of x will reverse our usual formula. In this case, the surface area will be

$$SA = \int_1^4 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy = \pi\int_1^4 \sqrt{4y+1}\,dy = \frac{\pi}{6}(4y+1)^{3/2}\Big|_1^4 = \frac{\pi}{6}(17^{3/2} - 5^{3/2})$$