

Calculus II

For the Exam...

- The exam will be about $1\frac{1}{2}$ times the length of a normal exam, and we have twice the amount of time to take it, so if you're well prepared, then time should not be an issue.
- As a reminder- If you do well on the final, then your lowest exam score will be replaced by the average of it and the final, so try your best!
- No calculators will be allowed, and no notes beyond the table of formulas and hints that were given to you in class.
- Final exam times:
 - Section A (9 MWF): Wed, Dec 17th, 2-4 PM
 - Section B (10 MWF): Tue, Dec 16th, 9-11 AM
- If you are free during those times, you may switch sections for the final exam- Please let me know a day or two in advance so I know how many copies I'll need and where you'll be.

The Integral in Theory

- The definition of the definite integral.
 - Write an integral from a Riemann sum, and a Riemann sum from an integral.
- Interpret the integral in terms of geometry (area of a circle or triangle, for example)
- The Fundamental Theorem of Calculus, Part I.

The primary condition is that the integrand, $f(x)$ is continuous on $[a, b]$. If

$$g(x) = \int_a^x f(t) dt$$

then g is continuous and differentiable, and $g'(x) = f(x)$.

Corollary:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

NOTE: The function g is a particular antiderivative- It is the antiderivative of f that so that $g(a) = 0$.

- The Fundamental Theorem of Calculus, Part II. The main computational tool of Calculus: If F is any antiderivative of the continuous function f ,

$$\int_a^b f(x) dx = F(b) - F(a)$$

- Understand the difference in notation: $\int f(x) dx$ $\int_a^x f(t) dt$ $\int_a^b f(x) dx$
- Understand the difference in notation: $\int_a^b \frac{d}{dx} f(x) dx$ $\frac{d}{dx} \int_a^x f(t) dt$ $\frac{d}{dx} \int_a^b f(x) dx$

- The Mean Value Theorem for Integrals. The average value of f is attained at some c in $[a, b]$. That is, if f is continuous on $[a, b]$, then there is a c in the interval so that:

$$f_{\text{avg}} = f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

- The improper integral is approximated by a definite integral, and is defined by taking the limit. For example,

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

NOTE: We need to recall techniques for computing a limit. For example, (i) algebraically simplify, (b) divide by x^n for some n , (c) l'Hospital's rule.

The Integral in Practice

We had several methods to evaluate an integral:

- Using geometry.
- u, du , or Substitution (Backwards Chain Rule)
- u, dv , or Integration by Parts (be able to use the tabular form of this)
- Partial Fractions. Also, be able to integrate something of the form $\int \frac{ax+b}{x^2+c} dx$
- Powers of sine and cosine. Remember the formulas for $\sin^2(x)$ and $\cos^2(x)$, and the main “trick” is to reserve something to get a substitution.
- Trigonometric substitution and the use of reference triangles.

Primarily, understand what we can substitute in each case using a trig identity:

$$a^2 - u^2, \quad u^2 + a^2 \quad u^2 - a^2$$

For example, in the first case, we substitute $u = a \sin(\theta)$, the expression simplifies to $a^2 \cos^2(\theta)$ (a perfect square).

- The table of integrals can be used (See the handout) as well.

Applications of the Integral

- Be able to compute the volume of a solid of revolution using disks, washers and shells. Let w be either x or y , depending on how the functions are defined. Then:

$$\int_a^b \pi R^2 dw \quad \pi \int_a^b (R^2 - r^2) dw \quad \int_a^b 2\pi r h dw$$

- Be able to compute the arc length or surface area (formulas provided).

Sequences to Series to Power Series to Taylor Series

Note the evolution of our notation in these sections:

$$\{a_n\}_{n=1}^{\infty}, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{k=1}^{\infty} c_k(x-a)^k, \quad \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Sequences:
 - What is a sequence?
 - Be able to determine if a sequence converges or diverges (Monotonic Sequence Theorem can be used, l'Hospital's rule, divide by an appropriate quantity, etc.)

- Series: $\sum_{n=1}^{\infty} a_n$

- Template series: Geometric Series (and the formula for the sum of a geometric series), p -series, harmonic series, alternating harmonic series.
- Convergence of the Series:
 - * Test for divergence.
 - * (For positive series) The direct ($a_n \leq b_n$) and limit comparison ($\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$) tests.
 - * (For positive series) The integral test, where $f(n) = a_n$ - We integrate $f(x)$.
 - * (For abs convergence) The Ratio Test and Root Tests. The Ratio Test is by far the most widely used test:

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

- * Check conditional convergence last: Alternating Series Test.
(The series has terms with alternating signs, the (abs value of the) terms are decreasing and the limit is zero).

- Power Series: $\sum_{k=1}^{\infty} c_k(x-a)^k$

- We have one of three choices for convergence. The series converges: (i) Only at $x = a$, (ii) for all x , or (iii) for $|x-a| < \rho$, and diverges for $|x-a| > \rho$. We say that ρ is the radius of convergence.
- Convergence is usually determined by the Ratio Test. We must check the endpoints of the interval separately (which gives the *interval of convergence*).
- Be able to get new series from a given series by differentiation or integration. We should also be able to compute a limit using a power series.

- Taylor Series: $\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ or Maclaurin: $\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

- Construct a Taylor series for an *analytic* function f based at $x = a$ (or a Maclaurin series, which is a Taylor series based at $a = 0$).
- Template series: e^x , $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$
- Find the sum of a series by recognizing it as a familiar Taylor series.

- Remainder for a Taylor series (using $T_n(x)$ to approximate $f(x)$) is (for some c between x and a):

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Binomial Theorem (and Binomial Coefficients)

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Template Series

$$e^x, \quad \sin(x), \quad \cos(x), \quad \frac{1}{1-x}$$