Solutions to the 2nd review

1. Limits:

(a) $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = 6.$

(b) $\lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{x\to 0} \left(\frac{\sin(5x)}{5x}\right) \cdot 5 = 5.$

In this problem, we use the fact that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$

(c) We can use l'Hospital's rule (note that the fraction starts with 0/0):

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

(d) Divide numerator and denominator by x^3: $\lim_{x\to\infty} \frac{2-5/x^2+1/x^3}{1+4/x} = 2.$

(e) Use l'Hospital's rule:

$\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \lim_{x\to 0} \frac{\frac{2}{1+2x}}{1} = 2.$

2. Continuity at $x = 2$ the left and right sides of the function match up.

$$\lim_{x\to 2^-} x^2 = 2^2 \qquad \lim_{x\to 2^+} mx + b = 2m + b$$

Therefore, $4 = 2m + b$.

Differentiability requires matching derivatives: $f'_-(2) = 4$ and $f'_+(2) = m$, so $m = 4$. Then $4 = 2(4) + b \Rightarrow b = -4$.

3. You could compute it directly using the quotient rule, or you can re-write as a product: $y = x^2 \sin x\, e^{-x}$. Then

$$\begin{aligned} y' &= e^{-x} \frac{d}{dx}(x^2 \sin x) + x^2 \sin x \frac{d}{dx}(e^{-x}) \\ &= e^{-x}(2x \sin x + x^2 \cos x) - x^2 \sin x\, e^{-x} \\ &= e^{-x}\Big(2x \sin x + x^2 \cos x - x^2 \sin x\Big). \end{aligned}$$

4. $f'(x) = \frac{2x}{x^2+1}$, so $f(1) = \ln 2$ and $f'(1) = 1$. Tangent line: $y - \ln 2 = 1(x-1)$, i.e. $y = x - 1 + \ln 2$.

5. Compute the second derivative $f''(x)$ for $f(x) = \frac{1}{x^2+1}$.

First,

$$f'(x) = -\frac{2x}{(x^2+1)^2}.$$

Then

$$f''(x) = -2\,(x^2+1)^{-2} + (-2x)\cdot(-2)(x^2+1)^{-3}(2x) = -\frac{2}{(x^2+1)^2} + \frac{8x^2}{(x^2+1)^3}.$$

With a common denominator $(x^2+1)^3$, $f''(x) = \frac{-2(x^2+1)+8x^2}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3} = \frac{2(3x^2-1)}{(x^2+1)^3}$.

6. This question uses the relationship between a function and it's inverse- That is, if the point (a, b) is on the graph of $f(x)$ and $f'(a) = m$, then (b, a) is on the graph of the inverse, with $(f^{(-1)})'(b) = \frac{1}{m}$.

We want $(f^{-1})'(1)$. By the relationship above,

$$(f^{-1})'(1) = \frac{1}{f'(x)}, \qquad \text{where } f(x) = 1$$

Looking at $x^3 + x + 1 = 1$, we observe that $x = 0$. Therefore, for $f(x)$, we are at the point $(0, 1)$. Further, $f'(x) = 3x^2 + 1$, so $f'(0) = 1$.

Finally, $\left(f^{(-1)}\right)'(1) = \frac{1}{1} = 1$.