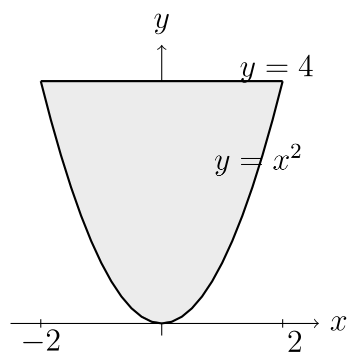
Calculus II Handout: Techniques of Integration

Solids of Revolution, Trigonometric Substitution, and Partial Fractions

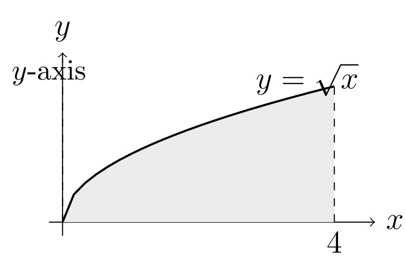
Part I. Solids of Revolution (Setup Only)

For each problem, *set up (do not evaluate)* the volume integral(s). Small sketches are included to guide your setup.

1. **Region bounded by** $y = x^2$ **and** $y = 4$; revolve about the x**-axis**.
   (a) Set up using **washers/disks**.
   (b) Set up using **shells**.



2. **Region bounded by** $y = \sqrt{x}$, $y = 0$, $x = 4$; revolve about the y**-axis**.
   (a) Use the **shell method**.
   (b) Sketch the region and indicate a representative shell.

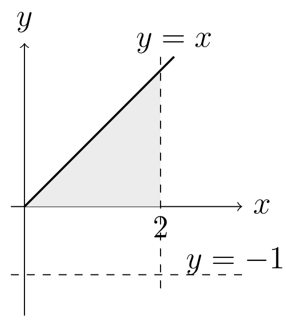


3. Same region as before, but rotate around the vertical line $x = 5$.

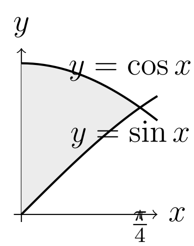
4. **Region bounded by** $y = x$, $y = 0$, $x = 2$; revolve about the line $y = -1$.

   (a) Set up using **washers**.

   (b) Clearly indicate the **outer** and **inner** radii.



5. **Region between** $y = \cos x$ **and** $y = \sin x$ on $[0, \pi/4]$; revolve about the x-**axis**. Use **washers**.



Part II. Trigonometric Substitution (Setup Only)

Hints:

For	Try
$a^2 - u^2$	$u = a \cdot \sin(\theta)$
$u^2 + a^2$	$u = a \cdot \tan(\theta)$
$u^2 - a^2$	$u = a \cdot \sec(\theta)$

For each integral, make the appropriate trignometric substitution and simplify your answer (but you do not need to integrate):

1. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$

2. $\displaystyle\int \frac{x^2}{\sqrt{x^2 + 16}}\, dx$

3. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 1}}$

4. $\displaystyle\int \sqrt{4 + 3x^2}\, dx$

Other similar types

After using partial fraction decomposition, you may end up with the following integrals to compute. See if you can compute them.

5. $\displaystyle\int \frac{1}{x - 3} + \frac{x + 1}{x^2 + 1}\, dx =$

6. $\displaystyle\int \frac{2}{x + 1} + \frac{x + 3}{(x + 1)^2} =$

7. $\displaystyle\int \frac{2}{x} + \frac{3}{x^2} + \frac{x + 2}{x^2 + 2} =$

Part III. Partial Fractions

Review: Long division

Perform the long division (if necessary) as a first step to Partial Fraction Decomposition.

1. $\dfrac{x+2}{x-5} =$

2. $\dfrac{2x^2+1}{x^2+3x+2} =$

3. $\dfrac{x^2+x+1}{x^2+1} =$

Find the coefficients

In the following, the decomposition has been given to you. Find the coefficients.

4. $\dfrac{3}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1}$

5. $\dfrac{x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

6. $\dfrac{x+2}{(x-1)^2(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+3}$

Partial Fractions (set up only)

For each expression, write down what the partial fraction decomposition would be (using constants A, B, C, etc). Do NOT solve for the coefficients. For example,

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

7. $\dfrac{3x+5}{x^3+2x^2} =$

8. $\dfrac{x^2+1}{x^3-x} =$

9. $\dfrac{1}{(x-1)^3(x^2+1)} =$

10. $\dfrac{x^2+5}{x^2+x}\,dx =$

Solutions

Part I. Solids of Revolution

(1) Region between $y = x^2$ and $y = 4$; about the x-axis.
(a) Washers: Outer radius $R = 4$, inner radius $r = x^2$; $x \in [-2, 2]$:

$$V = \pi \int_{-2}^{2} \left(R^2 - r^2\right) dx = \pi \int_{-2}^{2} \left(16 - x^4\right) dx.$$

(b) Shells (horizontal): $y \in [0, 4]$, radius $=\ y$, height $=\ (\sqrt{y} - (-\sqrt{y})) = 2\sqrt{y}$:

$$V = 2\pi \int_{0}^{4} \big(\text{radius}\big)\big(\text{height}\big)\, dy = 2\pi \int_{0}^{4} y \cdot 2\sqrt{y}\, dy = 4\pi \int_{0}^{4} y^{3/2}\, dy.$$

(2) About the y-axis; **shells** with vertical strips: radius $= x$, height $= \sqrt{x}$, $x \in [0, 4]$:

$$V = 2\pi \int_{0}^{4} x\sqrt{x}\, dx = 2\pi \int_{0}^{4} x^{3/2}\, dx.$$

(3) About $x = 5$: radius $= 5 - x$ and height is $\sqrt{x}$:

$$2\pi \int_{0}^{4} (5 - x)\sqrt{x}\, dx$$

(4) About $y = -1$; **washers** with vertical slices on $x \in [0, 2]$.
Outer radius $R(x) = \text{dist}(y = x,\ y = -1) = x + 1$; inner radius $r(x) = \text{dist}(y = 0,\ y = -1) = 1$:

$$V = \pi \int_{0}^{2} \Big[(x + 1)^2 - 1^2\Big]\, dx.$$

(5) About the x-axis; **washers** on $x \in [0, \pi/4]$. Since $\cos x \geq \sin x$ on this interval:

$$V = \pi \int_{0}^{\pi/4} \Big[(\cos x)^2 - (\sin x)^2\Big]\, dx.$$

Part II. Trigonometric Substitution solutions

1. Let $x = 3\sin\theta,\ dx = 3\cos\theta\, d\theta$:

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \int d\theta.$$

2. Let $x = 4\tan\theta,\ dx = 4\sec^2\theta\, d\theta$; $\sqrt{x^2+16} = 4\sec\theta$:

$$\int \frac{x^2}{\sqrt{x^2+16}}\, dx = \int \frac{16\tan^2\theta}{4\sec\theta}\,(4\sec^2\theta\, d\theta) = 16\int \tan^2\theta\ \sec\theta\, d\theta.$$

3. Let $x = \sec\theta,\ dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \tan\theta$:

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\ \tan\theta} = \int \cos\theta\, d\theta.$$

4. Let $x = 2\tan\theta,\ dx = 2\sec^2\theta\, d\theta$, $\sqrt{4+x^2} = 2\sec\theta$:

$$\int \sqrt{4+x^2}\, dx = \int \big(2\sec\theta\big)\big(2\sec^2\theta\, d\theta\big) = 4\int \sec^3\theta\, d\theta.$$

5. $$\int \left(\frac{1}{x-3} + \frac{x+1}{x^2+1}\right) dx = \int \frac{1}{x-3}\, dx + \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx$$

   The first integral is ready, the second integral can be done by substitution ($u = x^2+1$), and the third integral can be done by trig substitution generally, but it's straightforward this time.

$$= \ln|x-3| + \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C.$$

6. Let $u = x+1$. Then

$$\int \left(\frac{2}{x+1} + \frac{x+3}{(x+1)^2}\right) dx = \int \frac{2}{x+1}\, dx + \int \frac{x+3}{(x+1)^2}\, dx$$

   The first integral is ready. For the second, we can either do some algebra so that:

$$\frac{x+3}{(x+1)^2} = \frac{(x+1)+2}{(x+1)^2} = \frac{1}{x+1} + \frac{2}{(x+1)^2}$$

   Then use u, du substitution to integrate that second expression. Alternatively, we can just go ahead and use u, du right off the bat, with $u = x+1$:

$$\frac{x+3}{(x+1)^2} = \frac{u+2}{u^2} = \frac{1}{u} + \frac{2}{u^2}$$

In either case, for that second integral we get:

$$= \int \left(\frac{1}{u} + \frac{2}{u^2} \right) du == \ln|x+1| - \frac{2}{x+1},$$

so putting it all together,

$$2\ln|x+1| + \ln|x+1| - \frac{2}{x+1} + C = \boxed{3\ln|x+1| - \frac{2}{x+1} + C}.$$

7.

$$\int \left(\frac{2}{x} + \frac{3}{x^2} + \frac{x+2}{x^2+2} \right) dx = \int \frac{2}{x}\, dx + \int \frac{3}{x^2}\, dx + \int \frac{x}{x^2+2}\, dx + \int \frac{2}{x^2+2}\, dx.$$

Compute each term:

$$\int \frac{2}{x}\, dx = 2\ln|x|, \qquad \int \frac{3}{x^2}\, dx = -\frac{3}{x}, \qquad \int \frac{x}{x^2+2}\, dx = \tfrac{1}{2}\ln(x^2+2),$$

and using $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$ with $a = \sqrt{2}$,

$$\int \frac{2}{x^2+2}\, dx = \sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right).$$

Therefore

$$\boxed{2\ln|x| - \frac{3}{x} + \tfrac{1}{2}\ln(x^2+2) + \sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right) + C}.$$

Part III. Partial Fractions

Long Division

1.

$$\frac{x+2}{x-5} = \frac{(x-5)+7}{x-5} = 1 + \frac{7}{x-5}.$$

You could actually do long division:

$$\begin{array}{r} 1 \\ x-5 \overline{\smash{)}\, x+2} \\ -(x-5) \\ \hline 7 \end{array} \quad \Rightarrow \quad \frac{x+2}{x-5} = 1 + \frac{7}{x-5}$$

2.

$$\frac{2x^2+1}{x^2+3x+2} \quad \Rightarrow \quad \begin{array}{r} 2 \\ x^2+3x+2 \overline{\smash{)}\, 2x^2+0x+1} \\ -(2x^2+6x+4) \\ \hline -6x-3 \end{array} \quad \Rightarrow \quad \frac{2x^2+1}{x^2+3x+2} = 2 - \frac{6x3}{x^2+3x+2}.$$

We can stop there for the answer, but if we wanted to continue with Partial Fractions, we would factor the denominator

$$x^2 + 3x + 2 = (x+1)(x+2), \qquad \frac{-6x-3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

and so on.

3.
$$\frac{x^2+x+1}{x^2+1} = \frac{(x^2+1)+x}{x^2+1} == 1 + \frac{x}{x^2+1}.$$

Or you should get the same answer using long division.

Find the coefficients

4. $\dfrac{3x+5}{x^2+2x} = \dfrac{A}{x} + \dfrac{B}{x+2}.$

$$3x + 5 = A(x+2) + Bx \;\Rightarrow\; \begin{cases} A + B = 3, \\ 2A = 5 \end{cases} \quad \Rightarrow\; A = \tfrac{5}{2},\; B = \tfrac{1}{2}.$$

We can stop there for our answer, but notice that integrating our expression using partial fractions would then be:

$$\int \frac{3x+5}{x(x+2)}\,dx = \frac{5}{2}\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x+2} = \frac{5}{2}\ln|x| + \frac{1}{2}\ln|x+2| + C$$

5. Factor $x^3 - x = x(x-1)(x+1)$ and write

$$\frac{x^2+1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Matching coefficients:

$$x^2 + 1 = (A+B+C)x^2 + (B-C)x - A \;\Rightarrow\; A = -1,\; B = 1,\; C = 1.$$

We can stop there for our answer, but notice that integrating the expression using partial fractions would then be:

$$\int \frac{x^2+1}{x^3-x}\,dx = \int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\,dx = -\ln|x| + \ln|x-1| + \ln|x+1| + C$$

6. $\dfrac{1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} \Rightarrow A = \tfrac{1}{3},\; B = -\tfrac{1}{3}.$

Partial Fractions, Set up only

7. $\dfrac{3x+5}{x^3+2x^2} = \dfrac{3x+5}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2} = \dfrac{Ax+B}{x^2} + \dfrac{C}{x+2}$

8. $\dfrac{x^2+1}{x^3-x} = \dfrac{x^2+1}{x(x^2-1)} = \dfrac{x^2+1}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

9. $\dfrac{1}{(x-1)^3(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{Dx+E}{x^2+1}$

10. $\dfrac{x^2+5}{x^2+x}\,dx$ Long division first, or some tricky algebra:

$$\frac{x^2+5}{x^2+x} = \frac{(x^2+x)+(-x+5)}{x^2+x} = 1 - \frac{x-5}{x^2+x}$$

So now, just the fraction $\frac{x-5}{x^2+x}$ would be expanded:

$$\frac{x-5}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$