

# Calculus II

## For the Exam...

- The exam will be about  $1\frac{1}{2}$  times the length of a normal exam, and we have twice the amount of time to take it.
- As a reminder- If you do well on the final, then your lowest exam score will be replaced by the average of it and the final, so try your best!
- No calculators will be allowed, and no notes. However, I will provide the table of integrals and the sum formulas for  $\sum i^2$  and  $\sum i^3$ .
- If you are free during those times, you may switch sections for the final exam- Please let me know a day or two in advance so I know how many copies I'll need and where you'll be.

## The Integral in Theory

- How to write a “proof by induction” (and do a proof by induction for some basic statements).
- The definition of the definite integral.
  - Write an integral from a Riemann sum.
  - Write a Riemann sum from an integral.
- Interpret the integral in terms of geometry.
- The Fundamental Theorem of Calculus, Part I.

This applies to a function  $f$  that is continuous on  $[a, b]$ .

- Sets  $g(x) = \int_a^x f(t) dt$  as a differentiable function of  $x$ .
- Says that this function is a particular antiderivative of  $f$ ,  $g(a) = 0$ .
- Be able to differentiate:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt$$

- The Fundamental Theorem of Calculus, Part II. The main computational tool of Calculus: If  $F$  is any antiderivative of the continuous function  $f$ ,

$$\int_a^b f(x) dx = F(b) - F(a)$$

- Understand the difference in notation:

$$\int f(x) dx \quad \int_a^x f(t) dt \quad \int_a^b f(x) dx$$

- Understand the difference in notation:

$$\int_a^b \frac{d}{dx} f(x) dx \quad \frac{d}{dx} \int_a^x f(t) dt \quad \frac{d}{dx} \int_a^b f(x) dx$$

- The Mean Value Theorem for Integrals. The average value of  $f$  is attained at some  $c$  in  $[a, b]$ . That is, if  $f$  is continuous on  $[a, b]$ , then there is a  $c$  in the interval so that:

$$f_{\text{avg}} = f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Or, the area of the rectangle whose length is  $b - a$  and whose height is  $f(c)$  is equal to the integral:

$$f(c)(b-a) = \int_a^b f(x) dx$$

- The improper integral (Types I and II) is approximated by a definite integral, and is defined by taking the limit. For example,

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

*NOTE: We need to recall techniques for computing a limit.* For example, (i) algebraically simplify, (b) divide by  $x^n$  for some  $n$ , (c) l'Hospital's rule.

## The Integral in Practice

We had several methods to evaluate an integral:

- Using geometry.
- $u, du$ , or Substitution (Backwards Chain Rule)
- $u, dv$ , or Integration by Parts (be able to use the tabular form of this)
- Partial Fractions. Also, be able to integrate something of the form  $\int \frac{ax+b}{x^2+c} dx$
- Powers of sine and cosine. In particular, remember the formulas for  $\sin^2(x)$  and  $\cos^2(x)$ .
- Trigonometric substitution and the use of reference triangles.

Note: Even though a table of integrals will be provided, there are some types of integrals we should still be able to do (See the review sheet for Exam 3).

- The table of integrals can be used as well.

## Applications of the Integral

- Be able to compute the volume of a solid of revolution using disks, washers and shells. Let  $w$  be either  $x$  or  $y$ , depending on how the functions are defined. Then:

$$\int_a^b \pi R^2 dw \quad \pi \int_a^b (R^2 - r^2) dw \quad \int_a^b 2\pi r h dw$$

- Be able to compute the arc length of a curve.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Then the arc length is  $\int_a^b ds$

- Be able to compute the surface area for something found by rotating a curve around a line. A shorthand for the formula is given by:

$$\int_a^b 2\pi r ds$$

## Sequences to Series to Power Series to Taylor Series

Note the evolution of our notation in these sections:

$$\{a_n\}_{n=1}^{\infty}, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{k=1}^{\infty} c_k(x-a)^k, \quad \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Sequences:

- What is a sequence?
- Be able to determine if a sequence converges or diverges (Monotonic Sequence Theorem can be used, l'Hospital's rule, divide by an appropriate quantity, etc.)

- Series:  $\sum_{n=1}^{\infty} a_n$

- Template series: Geometric Series (and the formula for the sum of a geometric series),  $p$ -series, harmonic series, alternating harmonic series.
- Convergence of the Series:
  - \* Test for divergence.
  - \* (For positive series) The direct ( $a_n \leq b_n$ ) and limit comparison ( $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ ) tests.
  - \* (For positive series) The integral test, where  $f(n) = a_n$ . We integrate  $f(x)$ .
  - \* (For abs convergence) The Ratio Test and Root Tests. The Ratio Test is by far the most widely used test:

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

- \* Check conditional convergence last: Alternating Series Test.  
(The series has terms with alternating signs, the (abs value of the) terms are decreasing and the limit is zero).

- Power Series:  $\sum_{k=1}^{\infty} c_k(x-a)^k$

- We have one of three choices for convergence. The series converges: (i) Only at  $x = a$ , (ii) for all  $x$ , or (iii) for  $|x - a| < R$ , and diverges for  $|x - a| > R$ . We say that  $R$  is the radius of convergence.
- Convergence is usually determined by the Ratio Test. We must check the endpoints of the interval separately (which gives the *interval of convergence*).
- Be able to get new series from a given series by differentiation or integration.

- Taylor Series:  $\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$  or Maclaurin:  $\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

- Construct a Taylor series for an *analytic* function  $f$  based at  $x = a$  (or a Maclaurin series, which is a Taylor series based at  $a = 0$ ).
- Template series:  $e^x$ ,  $\sin(x)$ ,  $\cos(x)$ ,  $\frac{1}{1-x}$
- Find the sum of a series by recognizing it as a familiar Taylor series.