

Review Questions, Calc I and App. E, 5.1-5.2

Here are the items from Calc I you should be familiar with:

Topic:	Section
Definition of continuity	Section 2.5, p. 122
Where is $f(x)$ continuous?	Theorem 7, p. 127
What is the domain of $f(x)$?	Section 1.1, p. 12
	Appendix A
$\lim_{x \rightarrow \infty} f(x)$	Section 2.6, p. 137
(horizontal asympt)	Examples 3, 4,5
Some antiderivatives	Section 4.10

Test your understanding by answering the following questions from this material!

1. If $f(x) = \sqrt{\frac{1-x^2}{x^2-4}}$, where is f continuous?

The function f is continuous where $\frac{1-x^2}{x^2-4} \geq 0$. Set up a sign chart:

$(1-x)$	+	+	+	-	-
$(1+x)$	-	-	+	+	+
$(x+2)$	-	+	+	+	+
$(x-2)$	-	-	-	-	+
	$x < -2$	$-2 < x < -1$	$-1 < x < 1$	$1 < x < 2$	$x > 2$

From this we see that the solution is $-2 < x \leq -1$, or $1 \leq x < 2$

2. Compute the limit, if it exists:

(a) $\lim_{n \rightarrow \infty} \frac{3n^3 + 5n^2 + 2n}{6n^3 + 2n + 1}$

Divide numerator and denominator by n^3 to get:

$$\lim_{n \rightarrow \infty} \frac{(3n^3 + 5n^2 + 2n)/n^3}{(6n^3 + 2n + 1)/n^3} = \lim_{n \rightarrow \infty} \frac{3 + \frac{5}{n} + \frac{2}{n^2}}{6 + \frac{2}{n^2} + \frac{1}{n^3}} = \frac{3}{6} = \frac{1}{2}$$

(b) $\lim_{n \rightarrow \infty} \frac{1 - \sqrt{n}}{1 + \sqrt{n}}$

Divide numerator and denominator by \sqrt{n} :

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}} - 1}{\frac{1}{\sqrt{n}} + 1} = -1$$

(c) $\lim_{n \rightarrow \infty} 6 + \frac{18}{5n^2} \cdot n(n+2)$

First, rewrite the expression and cancel an n :

$$\lim_{n \rightarrow \infty} 6 + \frac{18}{5} \cdot \frac{n+2}{n} = 6 + \frac{18}{5} = \frac{48}{5}$$

(d) $\lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - \sqrt{x^2 - 1}$ (This one is included for extra practice only)

This required a little trick we learned in Calc I: Multiply by the conjugate:

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \cdot \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} = \lim_{x \rightarrow \infty} \frac{x^2 + 1 - (x^2 - 1)}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$$

Which simplifies to:

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} = 0$$

3. The following questions give a Riemann Sum. What definite integral does each represent?

To solve each of these, we first get the interval length $b - a$, then try to get a “good” value of a . Notice from exercises 4 and 5 that we can use any a , but there may be a natural choice. Once this is determined, write down the i^{th} right endpoint, and figure out what $f(x)$ is.

$$(a) \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[\left(\frac{i}{n} \right)^3 + 1 \right]$$

Here, $b - a = 1$ from the $\frac{1}{n}$ term. Looking at the function $(i/n)^3 + 1$, it looks like $a = 0$ is a natural choice, because then the i^{th} right endpoint is (i/n) . Now, the function took (i/n) , cubed it, and added one. This gives $f(x) = x^3 + 1$. Altogether, the answer is:

$$\int_0^1 x^3 + 1 \, dx$$

$$(b) \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$$

In this case, $b - a = 3$. Inside the square root, I see $1 + \frac{3i}{n}$ which is the i^{th} right endpoint if $a = 1$. This gives $f(x) = \sqrt{x}$.

$$\int_1^4 \sqrt{x} \, dx$$

$$(c) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + (i/n)^2}$$

This one is similar to the first, $b - a = 1$ and it looks like $a = 0$. This gives the i^{th} right endpoint as i/n , so that $f(x) = \frac{1}{1+x^2}$.

$$\int_0^1 \frac{1}{1+x^2} \, dx$$

4. Do the last problem again, but now make your interval begin with $a = -1$.

For the first one, if $a = -1$, then $b = 0$, and the i^{th} right endpoint would be $-1 + \frac{i}{n}$. Now, what should we do to this expression to get $(i/n)^3 + 1$? First add 1, then cube that, then add 1 to that: $f(x) = (x + 1)^3 + 1$.

$$\int_{-1}^0 (x + 1)^3 + 1 \, dx$$

For the second one, if $a = -1$ then $b = 2$. The i^{th} right endpoint is $-1 + \frac{3i}{n}$. To get $\sqrt{1 + \frac{3i}{n}}$, we need to add 2, then take the square root: $f(x) = \sqrt{x + 2}$:

$$\int_{-1}^2 \sqrt{x + 2} \, dx$$

Similarly the last one is:

$$\int_{-1}^0 \frac{1}{1 + (x + 1)^2} \, dx$$

5. Do problem 3 again, but now make every interval $a = 2$, $b = 6$

In this problem, we are forcing $b - a = 4$. In the first problem then we need to rewrite the sum:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[\left(\frac{i}{n} \right)^3 + 1 \right] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{n} \cdot \frac{1}{4} \cdot \left[\left(\frac{i}{n} \right)^3 + 1 \right]$$

Now, I write the i^{th} right endpoint as $2 + \frac{4i}{n}$. I need a function that takes this expression and makes it:

$$\frac{1}{4} \cdot \left[\left(\frac{i}{n} \right)^3 + 1 \right]$$

First, I'll divide the i^{th} right endpoint by 4, which gives me $\frac{1}{2} + \frac{i}{n}$. Next, I'll subtract $\frac{1}{2}$, which leaves me with i/n , and it's clear how to finish. So,

$$f(x) = \frac{1}{4} \left(\left(\frac{x}{4} - \frac{1}{2} \right)^3 + 1 \right)$$

and the integral is:

$$\int_2^6 \frac{1}{4} \left(\left(\frac{x}{4} - \frac{1}{2} \right)^3 + 1 \right) dx$$

We proceed in a similar manner in the second and third cases. In the second case, $a = 2, b = 6$ means that I need to rewrite the sum as:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{n} \cdot \frac{3}{4} \cdot \sqrt{1 + \frac{3i}{n}}$$

Now, the i^{th} right endpoint can be written as $2 + \frac{4i}{n}$, but I need that second term to be $\frac{3i}{n}$, so I'll first multiply by $\frac{3}{4}$, which leaves me with $\frac{3}{2} + \frac{3i}{n}$. I need this to be $1 + \frac{3i}{n}$, so next I'll subtract $\frac{1}{2}$. Putting it all together now:

$$\int_2^6 \frac{3}{4} \sqrt{\frac{3}{4}x - \frac{1}{2}} dx$$

Finally, in the third case, again rewrite the sum:

$$\lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \frac{1}{4(1 + (i/n)^2)}$$

Rewrite the i^{th} right endpoint as $2 + \frac{4i}{n}$. As is the first case, we need $\frac{i}{n}$, so first divide by 4, leaving $\frac{1}{2} + \frac{i}{n}$, then subtract $\frac{1}{2}$, leaving $\frac{i}{n}$. This leaves us with:

$$\int_2^4 \frac{1}{4(1 + (\frac{x}{4} - \frac{1}{2})^2)} dx$$

6. Give the general antiderivative, $F(x)$, where $f(x)$ is given as:

(a) $f(x) = 2x + 5(1 - x^2)^{-1/2}$

$$F(x) = x^2 + 5 \sin^{-1}(x) + C$$

(b) $f(x) = \frac{3}{x^2} - \frac{5}{x^4} + \sqrt[3]{x^2} = 3x^{-2} - 5x^{-4} + x^{2/3}$

$$F(x) = -3x^{-1} + \frac{5}{3}x^{-3} + \frac{3}{5}x^{5/3} + C$$

(c) $f(x) = 4 - 3(1 + x^2)^{-1}, F(1) = 0$ (Give the specific antiderivative)

$$F(x) = 4x - 3 \tan^{-1}(x) + C, F(1) = 0 \Rightarrow 0 = 4 - 3 \frac{\pi}{4} + C$$

because $\tan^{-1}(1) = \frac{\pi}{4}$. Putting in this value of C :

$$F(x) = 4x - 3 \tan^{-1}(x) + \frac{3\pi - 16}{4}$$

7. Set up, but do not evaluate, the integral of the given function on the given interval *using the definition*: The definition we will use will be the limit of the Riemann Sum, using equally spaced rectangles and right endpoints.

(a) $f(x) = 1 + x, 2 \leq x \leq 3$

$$\int_2^3 1 + x \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(1 + \left(2 + \frac{i}{n} \right) \right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(3 + \frac{i}{n} \right)$$

(b) $f(x) = \cos(x), \pi \leq x \leq 2\pi$

$$\int_{\pi}^{2\pi} \cos(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{n} \left(\cos\left(\pi + \frac{\pi i}{n}\right) \right)$$

(c) $f(x) = 2 + x + 3x^2, -1 \leq x \leq 3$

$$\int_{-1}^3 2 + x + 3x^2 \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{n} \left(2 + \left(-1 + \frac{4i}{n}\right) + 3\left(-1 + \frac{4i}{n}\right)^2 \right)$$

8. Evaluate each limit. You may use the formulas:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(a) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n}\right)^2$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{i^2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$$

(b) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(\frac{i^3}{n^3} + 1\right)$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i^3}{n^3} + 1\right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n \frac{i^3}{n^3} + \sum_{i=1}^n 1\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{n^3} \sum_{i=1}^n i^3 + n\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{n^3} \left(\frac{n(n+1)}{2}\right)^2 + n\right) = \lim_{n \rightarrow \infty} \left(\frac{n^2(n+1)^2}{4n^4} + 1\right) = \frac{5}{4} \end{aligned}$$

(c) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{n} \left[\sum_{i=1}^n \left(\frac{2i}{n}\right)^3 + 5 \sum_{i=1}^n \left(\frac{2i}{n}\right)\right] &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{8}{n^3} \sum_{i=1}^n i^3 + \frac{10}{n} \sum_{i=1}^n i\right] = \\ \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{8}{n^3} \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{10}{n} \cdot \frac{n(n+1)}{2}\right] &= 4 + 10 = 14 \end{aligned}$$

9. Check your answers to the previous problem by first setting up, then evaluating, the definite integral that each limit represents.

(a) $\int_0^1 x^2 \, dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$

(b) $\int_0^1 x^3 + 1 \, dx = \frac{1}{4}x^4 + x \Big|_0^1 = \frac{5}{4}$

(c) $\int_0^2 x^3 + 5x \, dx = \frac{1}{4}x^4 + \frac{5}{2}x^2 \Big|_0^2 = \frac{16}{4} + \frac{5 \cdot 4}{2} = 4 + 10 = 14$